CHAPTER 12

INFINITE SERIES

12.1 The Sum of an Infinite Series

PREREQUISITES

1. Recall the concept of convergence and divergence for infinite sequences and improper integrals (Sections 11.3 and 11.4).
2. Recall how to compute a limit at infinity (Sections 1.2, 11.1 and 11.2).

PREREQUISITE QUIZ

1. Show that $\int_0^\infty [dx/(x^2 + 2)]$ converges.
2. Does the sequence $(-1)^n$ converge as $n \to \infty$? Explain.
3. Compute the following limits:
   (a) $\lim_{x\to\infty}(e^x/x^4)$
   (b) $\lim_{x\to\infty}(\cos x/x^2)$

GOALS

1. Be able to compute the sum of a geometric series.
2. Be able to use the i^{th} term test to show divergence.

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STUDY HINTS

1. Infinite series, partial sums, and sequences. Suppose we have a sequence of numbers. The sum of the first n numbers is called the nth partial sum. The partial sums form a sequence. An infinite series is the sum of an infinite number of terms. If the partial sums have a limit, it is the sum of the series.

2. Convergence vs. divergence. If a series has a finite sum, it converges; otherwise, it diverges. Convergence requires a finite sum; that is, the limit of the partial sums must exist. If the limit of the partial sums does not exist, the series diverges. For example, $1 - 1 + 1 - 1 + 1 - 1 + \ldots$ has partial sums alternating between 1 and 0. Thus, the limit of the partial sums does not exist, and the series diverges. This example shows that convergence implies that a series has a finite value, but divergence does not imply that a series tends to infinity.

3. Notation. a_i is the symbol used for the i^{th} term of a series.. S_i is the symbol denoting the i^{th} partial sum.

4. Properties of limits of sequences. Most of the box on p. 563 is common sense. Property 9 is mostly review from p. 542. If $|r| < 1$ or $|r| > 1$, you should understand the result from your work with exponents. If $r = 1$, then $\lim_{n\to\infty} r^n = 1$. If $r = -1$, r^n alternates between 1 and -1, so no limit exists.

5. Geometric series. You should memorize the fact that $\sum_{i=0}^{\infty} ar^i = a/(1 - r)$ (first term)/(1 - ratio), provided $|r| < 1$. The series begins at $i = 0$, so a is the first term.

6. Algebraic rules. Again, common sense should tell you the validity of these statements. Note that the statements apply only for convergent series.

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7. Important tails. Any changes which occur at the beginning of a series do not affect the convergence or divergence. What is important for convergence is the behavior as i approaches infinity. Note how this example applies to Example 7.

8. i^{th} term test. Divergence is guaranteed if a_i does not approach 0 as i approaches ∞. Note that a_i approaching 0 does not guarantee convergence. The harmonic series is a counterexample.

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. By definition, $S_n = \Sigma_{i=1}^{n} a_i$. Thus, $S_1 = 1/2$; $S_2 = 1/2 + 1/3 = 5/6$; $S_3 = 1/2 + 1/3 + 1/4 = 13/12$; $S_4 = 1/2 + 1/3 + 1/4 + 1/5 = 77/60$.

5. This is a geometric series, so we use $\Sigma_{i=0}^{\infty} ar^i = a/(1 - r)$. The formula may be used since $r = 1/7 < 1$. The first term is $a = 1$, so the sum is $1/(1 - 1/7) = 1/(6/7) = 7/6$.

9. During the first year, you draw out \$10,000. Next year, you draw out $(3/4)(\$10,000)$, then $(3/4)^2(\$10,000)$, then $(3/4)^3(\$10,000)$, and so forth. For an arbitrarily large life span, the total amount to be drawn out is $\Sigma_{i=0}^{\infty}(3/4)^i(10,000) = 10,000/(1 - 3/4) = \$40,000$.

13. The series is $\Sigma_{i=0}^{\infty}(2^{3i+4}/3^{2i+5}) = 2^4/3^5 + 2^7/3^7 + 2^{10}/3^9 + \ldots$. This is a geometric series beginning with $2^4/3^5$ and having a ratio $r = 2^3/3^2 = 8/9 < 1$. Thus, the sum is $(2^4/3^5)/(1 - 8/9) = 9(2^4/3^5) = 2^4/3^3 = 16/27$.

17. We have $(2^n + 3^n)/6^n = (2/6)^n + (3/6)^n = (1/3)^n + (1/2)^n$. Thus, $\Sigma_{n=1}^{\infty}[(2^n + 3^n)/6^n]$ is the sum of two geometric series and it equals $\Sigma_{n=1}^{\infty}(1/3)^n + \Sigma_{n=1}^{\infty}(1/2)^n = \Sigma_{n=0}^{\infty}(1/3)(1/3)^n + \Sigma_{n=0}^{\infty}(1/2)(1/2)^n = (1/3)/(1 - 1/3) + (1/2)/(1 - 1/2) = (1/3)/(2/3) + (1/2)/(1/2) = 3/2$.

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21. Using the method of Example 6, we have $\Sigma_{i=1}^{\infty}(1 + 1/2^i) = \Sigma_{i=1}^{\infty} 1^i + \Sigma_{i=1}^{\infty}(1/2)^i$. $\Sigma_{i=1}^{\infty}(1/2)^i$ converges since $r = 1/2$; however, $r = 1$ in $\Sigma_{i=1}^{\infty} 1^i$, so it diverges. Thus, the entire integral diverges.

25. Consider the terms $i/\sqrt{i-1}$. Divide by $\sqrt{i}/\sqrt{i}$ to get $\sqrt{i}/\sqrt{1 + 1/i}$. As $i \to \infty$, the denominator tends to 1 and the numerator tends to ∞ . Since $\lim_{i\to\infty} a_i \neq 0$, the i^{th} term test tells us the series diverges.

29. This series is equivalent to $1 + 1/2 + 1/2 + 1/2 + \ldots$. Since $\lim_{i\to\infty} a_i = 1/2 \neq 0$, the series diverges.

33. Let $a_i = 1$ and $b_i = -1$ for all i . Then $a_i + b_i = 0$ and $\Sigma_{i=0}^{\infty}(a_i + b_i) = 0$ converges, but both $\Sigma_{i=0}^{\infty} 1$ and $\Sigma_{i=0}^{\infty}(-1)$ diverge.

37. (a) Let t_k be the carriage transit time for trip k . Let d_k be the distance between the crews at the beginning of trip k . Then $t_{2n+1} = d_{2n+1}/(20 + 7)$ and $t_{2n+2} = d_{2n+2}/(20 + 5)$, since the speed is a sum of the riding and working speeds. Also, $d_{n+1} = d_n - (5 + 7)t_n = d_1(5 + 7)\Sigma_{i=1}^{n} t_i = 12(1 - \Sigma_{i=1}^{n} t_i)$. Therefore, $t_{2n+1} = (12/27)(1 - \Sigma_{i=1}^{2n} t_i)$, and $t_{2n+2} = (12/25)(1 - \Sigma_{i=1}^{2n+1} t_i) = (12/25)(1 - \Sigma_{i=1}^{2n} t_i - t_{2n+1}) = (12/25)[1 - \Sigma_{i=1}^{2n} t_i - (12/27)(1 - \Sigma_{i=1}^{2n} t_i)] = (12/25)(15/27)(1 - \Sigma_{i=1}^{2n} t_i) = (15/25)t_{2n+1}$. Similarly, $t_{2n+1} = (12/27)[1 - \Sigma_{i=1}^{2n-1} t_i - (12/25)(1 - \Sigma_{i=1}^{2n-1} t_i)] = (12/27) \times (13/25)(1 - \Sigma_{i=1}^{2n-1} t_i) = (13/27)t_{2n}$. Therefore, $t_{2n+1} = (13/27) \times (12/25)t_{2n-1} = r^n t_1 = r^n \cdot (12/27)$. Also, $t_{2n+2} = (15/25)t_{2n+1} = r^n \cdot (15/25)(12/27) = r^{n+1} \cdot (12/13)$.

(b) The total time for carriage travel is $\lim_{n\to\infty} \Sigma_{i=1}^{n} t_i = \lim_{n\to\infty}(\Sigma_{i=0}^{n} t_{2i+1} + \Sigma_{i=0}^{n} t_{2i+2}) = \lim_{n\to\infty}[\Sigma_{i=0}^{n} r^i \cdot (12/27) + \Sigma_{i=0}^{n} r^{i+1} \cdot (12/13)]$. Using the fact that $1 + r + r^2 + \ldots + r^n = (r^{n+1} - 1)/(r - 1)$, we get $\lim_{n\to\infty}(12/27)[(r^{n+1} - 1)/(r - 1)] + \lim_{n\to\infty} r \cdot (12/13) \cdot [(r^{n+2} - 1)/(r - 1)] =$

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37. (b) (continued)

$(12/27)[(-1)/(r - 1)] + r\cdot(12/13)[-1/(r - 1)]$ (since $|r| < 1$) = $[(12/27) + r\cdot(12/13)]/(1 - r) = [(12/27) + (12/27)\cdot(15/25)]/(1 - r) = (12/27)\cdot(40/25)/\{[25\cdot27 - (13\cdot15)]/25\cdot27\} = [(12\cdot40)/(25\cdot27)] \times [25\cdot27/480] = 1$.

SECTION QUIZ

1. (a) Discuss the convergence or divergence of $(1 + 0 - 1) + (1 + 0 - 1) + (1 + 0 - 1) + \ldots$.

   (b) What happens if the parentheses are removed from the series in part (a)?

2. The series $1 + 3/2 + 9/4 + 27/8 + \ldots$ is a geometric series with ratio $3/2$. Thus, the sum is $1/(1 - 3/2) = -2$. The sum of positive numbers can't be negative. What's wrong?

3. The sum of a geometric series is 5 and the first term of the series is 2 . Write the series in the form $\Sigma_{i=0}^{\infty} ar^i$.

4. Consider the series $\Sigma_{i=1}^{\infty} 3$.

   (a) Use the i^{th} term test to analyze the series.

   (b) Does part (a) tell you anything about the tenth partial sum? Explain.

5. Mindless Marvin, the mixed-up medical student, wanted to start one of his patients on a drug which would be used for the rest of her life. Mindless Marvin knew that drugs obeyed exponential decay in the body so that $A \exp(-kt)$ would be left after the first dose. After the second dose, $A[\exp(-kt) + \exp(-2kt)]$ is left. Even Mindless Marvin could see that after n doses, the amount in the body would $A[\exp(-kt) + \exp(-2kt) + \ldots + \exp(-nkt)]$. Seeing this, Mindless

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5. (continued)

Marvin runs to the pharmacist and asks, "Won't the drug level become infinite and kill the patient after several doses?" The pharmacist responds, "No, Mindless Marvin." Explain this if A, k, and t are constants and $|\exp(-kt)| < 1$.

ANSWERS TO PREREQUISITE QUIZ

1. $\int_0^\infty [dx/(x^2 + 2)] = \int_0^1 [dx/(x^2 + 2)] + \int_1^\infty [dx/(x^2 + 2)]$. $\int_0^1 [dx/(x^2 + 2)]$ is proper, so it converges. $\int_1^\infty [dx/(x^2 + 2)] \leq \int_1^\infty (dx/x^2) = -x^{-1}|_1^\infty = 1$. Therefore, the entire integral converges.

2. It diverges; it oscillates between -1 and 1.

3. (a) $+\infty$

   (b) 0

ANSWERS TO SECTION QUIZ

1. (a) The partials sums are $0, 0, 0, \ldots$. Thus, the series converges to zero.

   (b) Without the parentheses, the partial sums are $1, 0, 0, 1, 0, 0, \ldots$. Thus, the series diverges since the partial sums have no limit.

2. The formula $a/(1 - r)$ applies only if $|r| < 1$.

3. $\Sigma_{i=0}^{\infty} 2(3/5)^i$

4. (a) It diverges because $a_i = 3$ for all i and $\lim_{i\to\infty} a_i = 3 > 1$.

   (b) Partial sums always exist; here $S_{10} = 30$. The i^{th} term test only applies to infinite series, not to partial sums.

5. This is a geometric series where $r = \exp(-kt)$. Thus, the maximum drug level is $A \exp(-kt)/[1 - \exp(-kt)]$.

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12.2 The Comparison Test and Alternating Series

PREREQUISITES

1. Recall how to use the comparison test for studying improper integrals (Sections 11.3).
2. Recall that the harmonic series diverges (Section 12.1).

PREREQUISITE QUIZ

1. Discuss the convergence or divergence of $\Sigma_{i=5}^{\infty}[i/(i^2 - 4i)]$.
2. Show that $\int_1^{\infty}[1/(x^5 + x^3 + 2x^2)]\,dx$ converges.
3. Show that $\int_1^{\infty}[1/(\sqrt{x} + \sqrt[3]{x})]\,dx$ diverges.

GOALS

1. Be able to demonstrate convergence or divergence by using the comparison test.
2. Be able to show convergence or divergence by using the ratio comparison tests.
3. Be able to distinguish between absolute and conditional convergence.
4. Be able to apply the alternating series test to show convergence.

STUDY HINTS

1. Comparison test. You can only draw a conclusion if the terms of a series are smaller than a known convergent series or larger than a known divergent series. If a series is shown to be termwise larger than a convergent series, you have no information; it may converge or diverge. Similarly, showing that a series is termwise smaller than a divergent series tells you nothing. Example: $\Sigma_{i=1}^{\infty}(1/n^2)$ and $\Sigma_{i=1}^{\infty}(1/n)$ are both less than $\Sigma_{i=1}^{\infty}1$, but this comparison tells you nothing about the convergence

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1. (continued)

or divergence of $\Sigma_{i=1}^{\infty}(1/n^2)$ or $\Sigma_{i=1}^{\infty}(1/n)$. Note the use of absolute values in the test. All of the terms being compared must be positive.

2. Using the comparison test. The first step is to guess convergence or divergence by resemblance to a known series. If one wishes to show convergence, one can make the numerator larger and the denominator smaller. If the new series is less than a known convergent series, then the original series converges. As an example, consider the series $\Sigma_{i=1}^{\infty}[(4i^4 + i^3 - i^2)/(i^6 + 7i^8)]$. Note that as i becomes very large, the numerator may be made larger by increasing the exponent of positive terms or by deleting negative terms. Similarly, the denominator may be made smaller by deleting the positive terms with large exponents. Thus, $(4i^4 + i^3 - i^2)/(i^6 + 7i^8) \leq (4i^4 + i^4)/i^6 = 4/i^2$ for large i and so, the series in question converges. Similar techniques may be used if divergence is suspected.

3. Ratio convergence tests. These are usually easier to use than the comparison test. Convince yourself of their validity. If $\lim_{i\to\infty}(|a_i|/b_i) < \infty$, then eventually, $|a_i| < cb_i$, where c is some finite constant. Again, convergence is implied. Similar reasoning should convince you of the second part of the test.

4. Error estimates. Often, we wish to estimate the sum of a series; however, an estimate is useless if it is far from its true answer. Error analysis helps us to decide if an estimate is useful. Study Example 5 to see how upper bounds of errors are estimated.

5. Alternating series. All alternating series converge with error no greater than $|a_{n+1}|$ if the partial sum, $S_n = a_1 + a_2 + \ldots + a_n$, is the estimate of the sum. The three conditions in the top box on p. 573

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5. Alternating series (continued).

need to be shown before a series can be called alternating.

6. Absolute vs. conditional convergence. Absolute convergence means a series would still converge if all the minus signs were removed. For example, $\Sigma_{n=1}^{\infty}(-1)^n(3/5)^n$ converges absolutely as a geometric series. Conditional convergence means that the minus signs are necessary. For example, $\Sigma_{i=1}^{\infty}[(-1)^i/i]$ converges, but if the minus signs are removed, it would diverge.

7. Increasing sequence property. This is theoretical material and is optional for most classes. Ask your instructor. Simply stated, a sequence increasing toward an upper bound has a limit. No statement is made about its associated series.

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. For $i \geqslant 1$, $8/(3^i + 2) \leqslant 8/3^i$. Also, $\Sigma_{i=1}^{\infty}(8/3^i) = 8\Sigma_{i=1}^{\infty}(1/3)^i$, which is a convergent geometric series. Thus, $\Sigma_{i=1}^{\infty}[8/(3^i + 2)]$ converges by the comparison test.

5. For $i \geqslant 1$, $|a_i| = 1/(3^i + 2) \leqslant 1/3^i$ and $\Sigma_{i=1}^{\infty}(1/3)^i$ is a convergent geometric series. Thus, $\Sigma_{i=1}^{\infty}[(-1)^i/(3^i + 2)]$ converges by the comparison test.

9. For $i \geqslant 1$, $3/(2 + i) \geqslant 3/(2i + i) = 1/i$, and $\Sigma_{i=1}^{\infty}(1/i)$ is the divergent harmonic series. Thus, $\Sigma_{i=1}^{\infty}[3/(2 + i)]$ diverges by the comparison test.

13. For $n \geqslant 1$, $|a_n| = 3/(4^n + 2) \leqslant 3/4^n$ and $\Sigma_{n=1}^{\infty}(3/4^n) = 3\Sigma_{n=1}^{\infty}(1/4^n)$ is a convergent geometric series. Thus, $\Sigma_{n=1}^{\infty}[3/(4^n + 2)]$ converges by the comparison test.

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17. Note that $1/(3i + 1/i) > 1/(3i + 3) = (1/3)(1/(i + 1))$. Since $(1/3)\Sigma_{i=1}^{\infty}[1/(i + 1)]$ is a harmonic series, it diverges. Then, by comparison, $\Sigma_{i=1}^{\infty}[1/(3i + 1/i)]$ diverges also.

21. Let $a_i = [1 + (-1)^i]/(8i + 2^{i+1})$ and $b_i = 2/2^{i+1} = 1/2^i$. Since $1 + (-1)^i$ equals either 0 or 2 , $|1 + (-1)^i| \leq 2$. Since $8i + 2^{i+1} > 2^{i+1}$, $|a_i| = [1 + (-1)^i]/(8i + 2^{i+1}) < 2/2^{i+1} = b_i$. Therefore, since $\Sigma_{i=1}^{\infty}b_i$ converges, so does $\Sigma_{i=1}^{\infty}a_i$.

25. Let $a_i = 3i/2^i$ and $b_i = 3(3/4)^i$. Since $i < (3/2)^i$, we have $|a_i| = 3i/2^i < 3(3/4)^i = b_i$. Therefore, since $\Sigma_{i=1}^{\infty}b_i$ converges, so does $\Sigma_{i=1}^{\infty}a_i$.

29. Let $a_j = \sin j/2^j$ and $b_j = 1/2^j$. Since $|\sin j| \leq 1$ for all j, we have $|a_j| \leq 1/2^j = b_j$. Thus, since $\Sigma_{j=1}^{\infty} b_j$ converges as a geometric series, $\Sigma_{j=1}^{\infty} a_j$ also converges by comparison.

33. Let $a_n = 1/(2^n + 1)$ and $b_n = 1/2^n$. Then $\lim_{n\to\infty}(a_n/b_n) = \lim_{n\to\infty}[2^n/(2^n + 1)] = 1$. Therefore, since $\Sigma_{n=1}^{\infty}b_i$ converges, so does $\Sigma_{n=1}^{\infty}a_i$.

37. Let $a_n = (2^n - 1)/(5^n + 1) < (2/5)^n$. The error in approximating the series by $\Sigma_{n=1}^{k}a_n$ is less than or equal to $(2/5)^{k+1}/(1 - 2/5) = 3(2^{k+1})/5^k < 0.01$ when $n = 10$. Thus, $\Sigma_{n=1}^{10}a_n \approx 0.37$.

41. Note that $\lim_{k\to\infty}[k/(k + 1)] = \lim_{k\to\infty}[1/(1 + 1/k)] = 1 \neq 0$. Thus, $\Sigma_{k=1}^{\infty}[k/(k + 1)]$ diverges by the i^{th} term test.

45. Let $a_1 = 1$, and for $i > 1$, $a_i = (-1)^{i+1}(i - 1)/i$. Then $\lim_{i\to\infty}|a_i| = 1 \neq 0$. Therefore, $\Sigma_{i=1}^{\infty}a_i$ diverges, by the i^{th} term test.

49. The derivative of $\ln[(n + 1)/n]$ is $1/n(n + 1)$, so it is a decreasing function for $n > 1$. The signs alternate and $\lim_{n\to\infty}\ln[(n + 1)/n] = \ln(1) = 0$. Therefore, the series converges conditionally as an alternating series.

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49. (continued)

Absolutely, $\Sigma_{n=1}^{\infty} \ln[(n+1)/n] = \Sigma_{n=1}^{\infty} [\ln(n+1) - \ln(n)]$. Since this is a telescoping sum, $\Sigma_{n=1}^{\infty} = \lim_{n\to\infty} \ln(n+1) - \ln(1) = \infty$. Thus, the series $\Sigma_{n=1}^{\infty} (-1)^n \ln[(n+1)/n]$ only converges conditionally.

53. The error in estimating $\Sigma_{n=1}^{\infty} (1/5)^n$ is $(1/5)(1/5)^{n+1}/(1 - 1/5) = (1/4)(1/5)^{n+1}$. The error in estimating $\Sigma_{n=1}^{\infty} [(-1)^n/2n]$ is less than $1/2n$. Therefore, $(1/4)(1/5)^{n+1} + 1/2n < 1/50$ for $n \geq 26$, and $\Sigma_{n=1}^{\infty} [(-1)^n/2n + 1/5^n] \approx \Sigma_{n=1}^{26} [(-1)^n/2n + 1/5^n] \approx -0.087$.

57. (a) $a_1 = \sqrt{4} = 2$; $a_2 = \sqrt{4+2} = \sqrt{6}$; $a_3 = \sqrt{4+\sqrt{6}}$.

(b) $a_1 = 2$; $a_2 = 2.44949$; $a_3 = 2.53958$; $a_4 = 2.55726$; $a_5 = 2.56071$; $a_6 = 2.56139$; $a_7 = 2.56152$; $a_8 = 2.56155$; $a_9 = 2.56155$; $a_{10} = 2.56155$; $a_{11} = 2.56155$; $a_{12} = 2.56155$. We guess that $\lim_{n\to\infty} a_n = 2.56155...$.

61. This is an increasing sequence which is bounded above. $a_{n+1} = 1/2(n+1) - 1/[(n+1)+1] = 1/(2n+2) - 1/(n+2) = -n/[(2n+2) \times (n+2)]$. $a_n = (1-n)/(2n)(n+1)$, so $a_{n+1} \geq a_n$ means $-2n^2 \times (n+1) \geq (1-n)(2n+2)(n+2)$ or $-2n^3 - 2n^2 \geq -2n^3 - 4n^2 + 2n + 4$ or $2n^2 \geq 2n + 4$. This is true if $n \geq 2$, so the sequence is increasing. For positive n , $1 - n \leq 0$ and $(2n)(n+1) \geq 0$, so $a_n \leq 0$; therefore, a_n is bounded above by 0 .

65. To show that a_n is increasing, we will show that $a_{n+1} \geq a_n$ or $a_n/2 + \sqrt{a_n} \geq a_n$, which implies $\sqrt{a_n} \geq a_n/2$. Rearrangement and squaring implies $4a_n \geq a_n^2$. Thus, if $0 \leq a_n \leq 4$, then a_n is increasing. Since $a_0 = 1$, a_{n+1} is always a sum of positive terms, so $a_n \geq 0$. We will now show a_n is bounded above by 4 using mathematical induction. For $n = 0$, $a_1 = 1/2 + 1 = 3/2 < 4$, which is true. Now, assume the statement holds for $n - 1$, i.e., $a_{n-1} \leq 4$. Then, $a_n \leq 4/2 + \sqrt{4} = 4$; therefore, a_n is increasing and bounded

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65. (continued)

above by 4 . By the increasing sequence property, $\lim_{n\to\infty} a_n$ converges to a limit, ℓ . To find ℓ , we solve $\lim_{n\to\infty} a_{n+1} = \ell = \ell/2 + \sqrt{\ell} = a_n/2 + \sqrt{a_n}$. The solution is $\ell = \lim_{n\to\infty} a_n = 4$.

69. For any x and any n , $\phi(4^n - x) \leq 1$. Let $b_n = (3/4)^n$ and let $a_n = (3/4)^n \phi(4^n x)$. Then we have $|a_n| = (3/4)^n \phi(4^n x) \leq (3/4)^n = b_n$. Therefore, since $\Sigma_{n=0}^{\infty} b_n$ converges as a geometric series, so does $\Sigma_{n=0}^{\infty} a_n$ by the comparison test.

SECTION QUIZ

1. Let $\Sigma_{i=1}^{\infty} a_i$ be the series to be analyzed and let $\Sigma_{i=1}^{\infty} b_i$ be a series whose convergence or divergence is known. What does the comparison test say? (More than one may be correct.)

   (a) If $|a_i| < |b_i|$ and $|\Sigma_{i=1}^{\infty} b_i|$ converges, then $\Sigma_{i=1}^{\infty} a_i$ also converges.

   (b) If $|b_i| < |a_i|$ and $|\Sigma_{i=1}^{\infty} b_i|$ converges, then $\Sigma_{i=1}^{\infty} a_i$ also converges.

   (c) If $|a_i| > |b_i|$ and $|\Sigma_{i=1}^{\infty} b_i|$ diverges, then $\Sigma_{i=1}^{\infty} a_i$ also diverges.

   (d) If $|b_i| > |a_i|$ and $|\Sigma_{i=1}^{\infty} b_i|$ diverges, then $\Sigma_{i=1}^{\infty} a_i$ also diverges.

2. Does $3 + 2 + 1 + 1/2 + 1/3 + \ldots + 1/100$ converge? Explain.

3. Can $\Sigma_{n=10}^{\infty} [(-1)^n (n - 5)/n]$ be analyzed by the alternating series test? Explain.

4. Determine if the following series converge conditionally, absolutely, or not at all. Justify your answer.

   (a) $\Sigma_{n=3}^{\infty} [-1/(3^n - 45)]$

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4. (b) $\Sigma_{n=0}^{\infty}[(-1)^n(n^2 + 5)/(n^3 + 3n^2 + 1)]$

   (c) $\Sigma_{n=-\infty}^{-1}(2^n/n^2)$

   (d) $\Sigma_{n=0}^{\infty}(1/n(3/2)^n)$

5. Little Lisa was a very bad girl. She went out last night and slept today rather than doing her homework. Because Little Lisa has a little brain, she can only remember 1/4 of what she learned yesterday, 4/6 of what she learned 2 days ago and so forth, i.e., she retains $n^2/(2 + 2^n)$ of what she learned n days ago. Explain in terms of convergence or divergence of infinite series whether Little Lisa's little brain is capacity-limited, i.e., does she have a bird-sized brain?

ANSWERS TO PREREQUISITE QUIZ

1. For $i > 5$, $i/(i^2 - 4i) \geq i/i^2 = 1/i$. Thus, $\Sigma_{i=5}^{\infty}[i/(i^2 - 4i)]$ diverges by the comparison test and by the divergence of the harmonic series.

2. $\int_1^{\infty}[dx/(x^5 + x^3 + 2x^2)] \leq \int_1^{\infty}[dx/2x^2] = (-1/2x)\Big|_1^{\infty}$, which converges.

3. $\int_1^{\infty}[dx/(\sqrt{x} + \sqrt[3]{x})] \geq \int_1^{\infty}[dx/(\sqrt{x} + \sqrt{x})] = \sqrt{x}\Big|_1^{\infty}$, which diverges.

ANSWERS TO SECTION QUIZ

1. a and c

2. This is not an infinite series, so it does converge. If the series continued as $+1/101 + 1/102 + \ldots$, then it would diverge.

3. No, $\lim_{n\to\infty}[(n - 5)/n] = 1 \neq 0$.

4. (a) Absolute convergence; $1/(3^n - 45) < 1/2^n$ if $n \geq 4$, and $\Sigma_{i=4}^{\infty}(1/2^n)$ is a geometric series.

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4. (b) Converges conditionally; $(n^2 + 5)/(n^3 + 3n^2 + 1) \geqslant n^2/n^3 = 1/n$ and $\Sigma_{n=0}^{\infty}(1/n)$ is the harmonic series. The original series converges by the alternating series test.

(c) Absolute convergence; $\Sigma_{n=-\infty}^{-1}(2^n/n^2) = \Sigma_{n=1}^{\infty}(1/2^n n^2) < \Sigma_{n=0}^{\infty}(1/2^n)$, which converges as a geometric series.

(d) Diverges; the $n = 0$ term is infinite. Note that $\Sigma_{n=1}^{\infty}[1/n(3/2)^n]$ does converge.

5. Her brain is capacity-limited; $\Sigma_{n=1}^{\infty}[n^2/(2 + 2^n)] \leqslant \Sigma_{n=1}^{\infty}(n^2/2n)$, which converges by the ratio test.

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12.3 The Integral and Ratio Tests

PREREQUISITES

1. Recall how to evaluate improper integrals (Section 11.3).
2. Recall that the geometric series converges (Section 12.1).
3. Recall how to use the i^{th} term test to demonstrate divergence (Section 12.1).
4. Recall how to use the comparison test to show convergence or divergence (Section 12.2).
5. Recall how to find limits by using l'Hôpital's rule (Section 11.2).

PREREQUISITE QUIZ

1. Evaluate $\int_0^\infty (dx/e^x)$.
2. Compute $\Sigma_{i=1}^{\infty} (2/5)^i$.
3. Compute $\lim_{x\to 1}[(x^5 + x^4 - 2)/(x - 1)]$.
4. Compute $\lim_{x\to\infty}[(x^5 + x^4 - 2)/(3x^5 + 2x)]$.
5. Discuss the convergence or divergence of the following series:
   (a) $\Sigma_{n=1}^{\infty}[(n + 1)/2n]$
   (b) $\Sigma_{i=1}^{\infty}[(i^5 + i^4 + 2)/(i^3 + i)]$.

GOALS

1. Be able to use either the integral test, the p-series test, the ratio test, or the root test for determining the convergence or divergence of a series.
2. Be able to estimate the error when an infinite series is approximated by a partial sum .

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STUDY HINTS

1. Integral test. Just understand the conclusion. If the integral converges, so does the series. If the integral diverges, so does the series. Although the test asks you to integrate from 1 to ∞, 1 is just the arbitrary starting point. Again, we are concerned mostly with the tail. Be sure that the initial terms are finite. For example, $\Sigma_{i=0}^{\infty}(1/n^2)$ diverges since the $i = 0$ term is infinite.

2. p-series. This test follows directly from the integral test. Know this statement well. A good way to remember this statement is to realize that any exponent less than one makes the series larger than the harmonic series. In estimating the sum by S_N, the error is less than $1/(p - 1)N^{p-1}$.

3. Ratio test. Do not confuse this with the ratio comparison test in Section 12.2. The idea is to compare a term in the tail of the series with the preceding term. It makes sense that if the ratio is absolutely less than one, then the terms are getting smaller in geometric progression, so we expect absolute convergence (at least, intuitively). If the terms are getting larger, then the ratio is greater than one, and divergence occurs. Consider the harmonic series and the alternating harmonic series to note that the test fails if the ratio is one. The ratio test is easy to use if factorials occur. See Example 7. Remember, the magic number for the ratio test is one. An upper bound for the error estimate is $|a_N|r/(1 - r)$.

4. Root test. Essentially, the test uses the following facts: If $0 < a_i < 1$, then any root of a_i is less than one. If $a_i > 1$, then any of its roots is greater than one. This is not a proof that the test works, but it should help you to remember it. Again, we are only interested in the tailing terms for the convergence issue.

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SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. We need to evaluate $\int_1^\infty [x/(x^2 + 1)]dx$. By letting $u = x^2 + 1$, we get $(1/2)\int_2^\infty (du/u) = \lim_{b\to\infty}(\ln u/2)\Big|_2^b$. The integral is ∞ , so according to the integral test, $\Sigma_{i=1}^\infty [i/(i^2 + 1)]$ diverges.

5. Since $|\cos n| \leq 1$ for all n , we have $\Sigma_{n=1}^\infty |\cos n/n^2| \leq \Sigma_{n=1}^\infty |1/n^2|$. Now, $p = 2 > 1$, so $\Sigma_{n=1}^\infty (\cos n/n^2)$ converges (absolutely).

9. The error in estimating a p-series is $\Sigma_{n=N+1}^\infty (1/n^p)$, which is less than or equal to $1/(p - 1)N^{p-1}$. The error is $\Sigma_{n=N+1}^\infty (\cos n/n^3) \leq \Sigma_{n=N+1}^\infty (1/n^3)$ since $|\cos n| \leq 1$. Thus, we need $1/2N^2 < 0.05$, i.e., $N \geq 4$. Therefore, $\Sigma_{n=1}^\infty (\cos n/n^3) \approx 0.44$.

13. The ratio $|a_n/a_{n-1}|$ is $(2\sqrt{n}/3^n)\cdot(3^{n-1}/2\sqrt{n-1}) = (1/3)\sqrt{n/(n-1)}$. Thus, $\lim_{n\to\infty}|a_n/a_{n-1}| = 1/3$, which is less than 1 . Therefore, the series $\Sigma_{n=1}^\infty (2\sqrt{n}/3^n)$ converges.

17. The error is less than $|a_N|r/(1 - r)$, where $|a_n/a_{n-1}| < r < 1$. The ratio is $r = \pi^2/(2n)(2n + 1)$, so the error is less than $[\pi^{2N+1}/(2N + 1)!][\pi^2/(2N)(2N + 1)]/[((2N)(2N + 1) - \pi^2)/(2N)(2N + 1)] = \pi^{2N+3}/(4N^2 + 2N - \pi^2)(2N + 1)!$. The error estimate is approximately 0.013 for $N = 4$, so $\Sigma_{n=0}^\infty [\pi^{2n+1}/(2n + 1)!] \approx \Sigma_{n=0}^4 [\pi^{2n+1}/(2n + 1)!] = \pi + \pi^3/6 + \pi^5/120 + \pi^7/5040 + \pi^9/362880 \approx 11.54$.

21. $a_n = 3^n/n^n$, so $|a_n|^{1/n} = 3/n$, and the limit $\lim_{n\to\infty}|a_n|^{1/n}$ is 0 . Therefore, $\Sigma_{n=1}^\infty (3^n/n^n)$ converges by the root test.

25. The series $\Sigma_{i=1}^\infty (1/i^4)$ converges by the p-series test with $p = 4 > 1$.

29. When k is even, $\cos k\pi = 1 > 0$. When k is odd, $\cos k\pi = -1 < 0$. Therefore, the terms of $(\cos k\pi)/\ln k$ alternate in sign. Since $\ln(k + 1) > \ln k$, $1/\ln(k + 1) < 1/\ln k$. $\lim_{k\to\infty}|(\cos k\pi)/\ln k| = \lim_{k\to\infty}(1/\ln k) = 0$. Therefore, the absolute values of the terms of this

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29. (continued)

series decrease to 0 . Hence $\Sigma_{k=0}^{\infty}[(\cos k\pi)/\ln k]$ converges by the alternating series test.

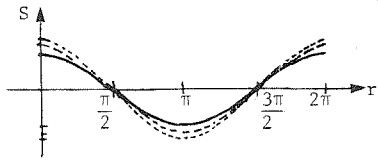
33. The series $\Sigma_{r=0}^{\infty}[2^r/(2^r + 3^r)] \geqslant \Sigma_{r=0}^{\infty}(2^r/3^r)$, which converges because of the root test: $\lim_{r\to\infty}(2^r/3^r)^{1/r} = \lim_{r\to\infty}(2/3) = 2/3 < 1$. By the comparison test, the original series converges.

37. If $\lim_{n\to\infty}|a_n|^{1/n} > 1$, then by the definition of the limit, there exists an N such that $|a_n|^{1/n} > 1$ for all $n > N$. Then, we also have $|a_n| > 1$ for all $n > N$. Since $\lim_{n\to\infty}|a_n| \neq 0$, the series $\Sigma_{n=1}^{\infty}a_n$ diverges by the i^{th} term test.

41. If $p > 1$, then $1/n^p \ln n < 1/n^p$ for $n \geqslant 3$ because $\ln n > 1$ for $n \geqslant 3$. Then, by the comparison test and the p-series test, $\Sigma_{n=2}^{\infty}(1/n^p \ln n)$ converges if $p > 1$. If $p = 1$, the integral test gives $\int_2^{\infty}(dx/x \ln x) = \int_{\ln 2}^{\infty}(du/u) = \lim_{b\to\infty} \ln u\Big|_{\ln 2}^{\infty}$, so the series diverges if $p = 1$. If $p < 1$, then $1/n^p \ln n > 1/n \ln n$, whose series diverges. Therefore, $\Sigma_{n=2}^{\infty}(1/n^p \ln n)$ converges for $p > 1$.

45. (a) Let $a_n = \{\cos[(2n + 1)^2 r]\}/(2n + 1)^4$; then $a_0 = \cos r$, $a_1 = \cos(9r)/3^4$, and $a_2 = \cos(25\, r)/5^4$.

(b)



$S_1(r)$ —— $S_1(0) = 1$

$S_2(r)$ - - $S_2(0) = 82/81 = 1.012$

$S_3(r)$ ··· $S_3(0) = 51331/50625 = 1.014$

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45. (c) Since $|\cos[(2n + 1)^2 r]| \leq 1$, $|a_n| \leq 1/(2n + 1)^4$. Let $f(x) = 1/(2x + 1)^4$. Since f is decreasing, the integral test may be applied. $\int_1^\infty f(x)dx = \lim_{b\to\infty}[1/6(2x + 1)^3]\Big|_1^b < \infty$. Therefore $\Sigma_{n=0}^{\infty}[1/(2n + 1)^4]$ converges, and so does $\Sigma_{n=0}^{\infty} a_n$.

SECTION QUIZ

1. Show that the ratio test cannot be used to analyze a p-series.

2. Discuss the convergence (conditional or absolute) or divergence of the following series:

   (a) $\Sigma_{n=-\infty}^{\infty}[\tan^{-1} n/(1 + n^2)]$

   (b) $\Sigma_{n=0}^{\infty}[(-1)^n(3r)!/(2n + 8)!]$

   (c) $\Sigma_{n=1}^{\infty} n^{-1-1/n}$

   (d) $\Sigma_{n=1}^{\infty}(\sqrt[n]{n} - 1)^n$

   (e) $\Sigma_{n=0}^{\infty}(4^n \cos n/2^n n!)$

   (f) $\Sigma_{n=-5}^{\infty}\left(1/\sqrt[3]{n^2 + 3n + 5}\right)$

   (g) $\Sigma_{n=7}^{\infty}(8^n/(n + 3)^n)$

   (h) $\Sigma_{n=-3}^{\infty}(n/\exp(n^2))$

3. A certain planet in a distant galaxy was on the verge of destruction from radioactivity. The survivors have protected themselves with a special serum which lines their entire body surface with lead. However, lead is not an unlimited resource on their planet. Suppose that the species is shrinking as an adaptive mechanism so that the total surface area of the n^{th} generation is proportional to $3^n/n!$. Will a finite amount of lead protect the population forever, i.e., is $\Sigma_{n=0}^{\infty}(3^n/n!)$ finite? Justify your answer.

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ANSWERS TO PREREQUISITE QUIZ

1. 1

2. 2/3

3. 9

4. 1/3

5. (a) $\lim_{n\to\infty}[(n + 1)/2n] = 1/2$, so the series diverges by the i^{th} term test.

   (b) $\lim_{i\to\infty}[(i^5 + i^4 + 2)/(i^3 + i)] = +\infty$, so the series diverges by the i^{th} term test.

ANSWERS TO SECTION QUIZ

1. $\Sigma_{i=1}^{\infty} i^p$ has ratio $(i - 1)^p/i^p = [(i - 1)/i]^p$, whose limit is one.

2. (a) Converges absolutely; use integral test.

   (b) Diverges; use i^{th} term test.

   (c) Converges absolutely; use root test.

   (d) Converges absolutely; use root test.

   (e) Converges absolutely; compare to $\Sigma_{n=0}^{\infty}(2^n/n!)$ and use ratio test.

   (f) Diverges; compare to $\Sigma_{n=1}^{\infty}(1/n^{2/3})$ and use p-series.

   (g) Converges; use root test.

   (h) Converges absolutely; use integral test.

3. Yes; by ratio test, $\lim_{n\to\infty}(3/n) = 0 < 1$.

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12.4 Power Series

PREREQUISITES

1. Recall how to apply the ratio test to demonstrate convergence or divergence (Section 12.3).
2. Recall how to apply the root test for analyzing infinite series (Section 12.3).
3. Recall how to differentiate and integrate polynomials (Chapters 1, 2, and 4).

PREREQUISITE QUIZ

1. Differentiate the following:
   (a) $x^5 - 3x^2 + x$
   (b) $6x^2 + x^4$
2. Perform the following integrations:
   (a) $\int_{-1}^{0}(x^3 + x)dx$
   (b) $\int(t^5 + t^2 + 3)dt$
3. Do the following series converge or diverge? Justify your answers.
   (a) $\Sigma_{n=1}^{\infty}(3/2n)^n$
   (b) $\Sigma_{n=1}^{\infty}(100n/n!)$

GOALS

1. Be able to find the radius of convergence of a power series.
2. Be able to differentiate, integrate, and algebraically manipulate convergent power series.

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STUDY HINTS

1. Power series. These series have the form $\Sigma_{i=0}^{\infty} a_i (x - x_0)^i$. They behave like regular functions in that they can be added, subtracted, multiplied, divided, differentiated, and integrated, but only in the regions where the power series converges.

2. Ratio test for power series. Learn the test for the general power series by replacing x with $x - x_0$. As with the ratio test of Section 12.3, you need to find the limit of a_i/a_{i-1}. For convergence, we want $[\lim_{i\to\infty}(a_i/a_{i-1})] \cdot |x - x_0| < 1$ or $|x - x_0| < 1/[\lim_{i\to\infty}(a_i/(a_{i-1}))]$. Thus, the radius of convergence, R, is the reciprocal of the limit. The series converges if $|x - x_0| < R$, i.e., $-R < x - x_0 < R$, i.e., $-R + x_0 < x < R + x_0$. To determine convergence at $-R + x_0$ and $R + x_0$, substitute into the original series and apply the tests presented earlier in the chapter. The ratio test will give a ratio of one at the endpoints, so it is mandatory to apply other tests to analyze the convergence or divergence at $x_0 \pm R$.

3. Root test for power series. As with the root test of Section 12.3, $\lim_{i\to\infty} |a_i|^{1/i}$ needs to be computed. Again, the radius of convergence is the reciprocal of the limit.

4. Differentiation and integration. In the differentiation formula, $(d/dx)\Sigma_{i=0}^{\infty} a_i (x - x_0)^i = \Sigma_{i=1}^{\infty} i a_i (x - x_0)^{i-1}$, we would like the derivative to converge at $x = x_0$. Thus, the exponent of $x - x_0$ should be nonnegative. Hence, in the differentiation process, the index on the right begins at the next higher integer in most cases. The index starting point does not change during the integration process. For both processes, the radius of convergence does not change.

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5. Algebraic manipulations. When two series are added, subtracted, or multiplied, the new radius of convergence is at least as big as the smaller of the original radii. The radius of convergence for a quotient must be determined after the new series is determined. Example 10 shows how tedious division can be.

6. Applications of power series. Starting with a known power series, manipulations may be performed to derive power series for new expressions. Differentiation and integration are commonly used. See Example 8.

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. Here, $\ell = \lim_{i\to\infty} |[2/(i+1)][i/2]| = 1$. Thus, the series converges if $|x| < 1$ and diverges if $|x| > 1$. If $x = 1$, $\Sigma_{i=0}^{\infty}[2/(i+1)] = 2 \times \Sigma_{i=1}^{\infty}(1/i)$ is a harmonic series; thus, it diverges. If $x = -1$, $2\Sigma_{i=0}^{\infty}[(-1)^i/(i+1)]$ is an alternating series; thus, it converges. Therefore, the series converges for $-1 \leq x < 1$.

5. We have $\ell = \lim_{i\to\infty} |[(5i+1)/i][(i-1)/(5(i-1)+1)]| = 1$, so $R = 1/\ell = 1$. Thus, the series converges absolutely for all x such that $|x-1| < 1$, i.e., $0 < x < 2$. Since $\lim_{i\to\infty}[(5i+1)/i] \neq 0$, the series diverges at the endpoints. Thus, $\Sigma_{i=1}^{\infty}[(5i+1)(x-1)^i/i]$ converges for $0 < x < 2$.

9. Here, $\ell = \lim_{n\to\infty} |[1/(2^n+4^n)][2^{n-1}+4^{n-1}]|$. Now, $2^{n-1}/(2^n+4^n) = 2^n/2(2^n+4^n) = 1/2(1+2^n)$ and $4^{n-1}/(2^n+4^n) = 4^n/4(2^n+4^n) = 1/4((1/2)^n+1)$; therefore, $\ell = \lim_{n\to\infty}[1/2(1+2^n) + 1/4((1/2)^n+1)] = 1/4$. Thus, the series converges if $|x| < 4$. If $x = -4$, then $x^n/(2^n+4^n) = (-4)^n/(2^n+4^n) = (-1)^n/((1/2)^n+1)$. The limit $\neq 0$, so the series diverges if $x = -4$. Similarly, the series diverges at $x = 4$. Therefore, $\Sigma_{n=1}^{\infty}[x^n/(2^n+4^n)]$ converges for $-4 < x < 4$.

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13. Here, $a_i = 5i/2^i$, so $\ell = \lim_{i\to\infty}(5i/2^i)(2^{i-1}/5(i-1)) = \lim_{i\to\infty}(5i/2 \cdot 5(i-1))$. Since the limit is $1/2$, the radius of convergence for $\Sigma_{i=1}^{\infty}(5ix^i/2^i)$ is 2. If $x = \pm 2$, the series is $\Sigma_{i=1}^{\infty}(-1)^i 5i$ or $\Sigma_{i=1}^{\infty}5i$, which both diverge. Therefore, $\Sigma_{i=1}^{\infty}(5ix^i/2^i)$ converges if $-2 < x < 2$.

17. For $a_n = (n^2 + n^3)/(1+n)^5$, $\ell = \lim_{n\to\infty}|a_n/a_{n-1}| = \lim_{n\to\infty}[(n^2+n^3)/(1+n)^5]\{n^5/[(n-1)^2 + (n-1)^3]\} = 1$ by l'Hôpital's rule. Thus, $R = 1/\ell = 1$. When $x = \pm 1$, $\Sigma_{n=0}^{\infty}|a_n x^n| = \Sigma_{n=0}^{\infty}[(n^2+n^3)/(1+n)^5] = \Sigma_{n=0}^{\infty} a_n$. Let $b_n = n^3/n^5$. Then $\lim_{n\to\infty}(|a_n|/b_n) = 1 < \infty$, and since $\Sigma_{n=0}^{\infty} b_n$ converges, $\Sigma_{n=0}^{\infty}a_n$ is absolutely convergent. Thus, $\Sigma_{n=0}^{\infty} a_n x^n$ converges for $x = 1$ and -1.

21. The $\lim_{n\to\infty}|(-1)^n n^n|^{1/n}$ is $\lim_{n\to\infty} n$, which is ∞. Therefore, the series converges only for $x = 0$, and the radius of convergence is 0.

25. (a) $\ell = \lim_{i\to\infty}[(i+1)/i] = 1$, so the radius of convergence is $R = 1/\ell = 1$.

(b) $\int_0^x f(t)dt = \int_0^x \Sigma_{i=1}^{\infty}(i+1)t^i dt = \Sigma_{i=1}^{\infty}\int_0^x (i+1)t^i dt = \Sigma_{i=1}^{\infty} x^{i+1}$.

(c) Using the geometric series $1/(1-x) = 1 + x + x^2 + \ldots$, note that $\Sigma_{i=1}^{\infty}x^{i+1} = \Sigma_{i=0}^{\infty}x^2 x^i = x^2/(1-x)$. Also $f(x) = (d/dx)\int_0^x f(t)dt = (d/dx)(x^2/(1-x)) = x(2-x)/(1-x)^2$ for $|x| < 1$.

(d) $2/2 + 3/4 + 4/8 + 5/16 + \ldots = \Sigma_{i=1}^{\infty}[(i+1)(1/2)^i] = f(1/2)$. Thus, using the result of part (c), the sum is $(1/2)(2 - 1/2)/(1 - 1/2)^2 = 3$.

29. Note that $(d/dx)\tan^{-1}x = 1/(1+x^2) = 1 - x^2 + x^4 - x^6 + \ldots$ by long division. So $\tan^{-1}x = \int_0^x[1/(1+t^2)]dt = x - x^3/3 + x^5/5 - x^7/7 + \ldots$, i.e., $\tan^{-1}x = \Sigma_{n=0}^{\infty}[(-1)^n x^{2n+1}/(2n+1)]$ and $(d/dx)\tan^{-1}x = \Sigma_{n=0}^{\infty}(-1)^n x^{2n}$.

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33. From Exercise 23, $\sin x = x - x^3/3! + x^5/5! - \ldots$. Therefore, we have $\sin^2 x = (x - x^3/3! + x^5/5! - \ldots)(x - x^3/3! + x^5/5! - \ldots) = x^2 - x^4/3 + (2/5! + 1/(3!)^2)x^6 - \ldots = x^2 - x^4/3 + 2x^6/45 \ldots$.

37. (a) Using long division with $\sin x = x - x^3/3! + x^5/5! - \ldots$ and $\cos x = 1 - x^2/2! + x^4/4! - x^6/6! + \ldots$, we get $\tan x = \sin x/\cos x = x + (1/3)x^3 + (2/15)x^5 + \ldots$.

(b) From the result of part (a) and differentiating term by term, we get $\sec^2 x = (d/dx)\tan x = 1 + x^2 + (2/3)x^4 + \ldots$.

(c) Using the result of part (b) and long division, we get $1/\sec^2 x = 1/[1 + x^2 - (2/3)x^4 + \ldots] = 1 - x^2 + (1/3)x^4 - \ldots$.

41. Let $f(x) = \sum_{i=0}^{\infty} a_i x^i$ have radius of convergence R_f . Thus, if $|x_0| < R$, this series converges. By the theorem, $\sqrt[i]{|a_i|} < 1/|x_0|$ for $i \geq N$. Now $\sqrt[i]{2} \to 1$ as $i \to \infty$ (for example, write $\sqrt[i]{i} = i^{1/i}$ and use l'Hôpital's rule, as in Example 7(a), p. 525 to show that $\lim_{x\to\infty} x^{1/x} = 1$). Thus, $\sqrt[i]{i}\sqrt[i]{|a_i|} < 1/|x_0|$ if i is large enough. By the root test, $\sum_{i=1}^{\infty} i a_i x_0^i$ converges. Thus $g(x_0) = \sum_{i=1}^{\infty} i a_i x_0^{i-1}$ converges as well. Therefore, $g(x)$ has radius of convergence $R_g \geq R_f$. A similar argument, interchanging f and g shows that $R_f \geq R_g$, so $R_f = R_g$. Likewise, one shows that $R_f = R_h$.

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45. Let $|x| < R$. From Exercise 44, $\int_0^x g(t)dt = f(x)$. Therefore, using the alternative version of the fundamental theorem of calculus, $f'(x) = g(x)$.

SECTION QUIZ

1. Suppose $f(x) = \Sigma_{i=0}^{\infty}(2/3^i)x^i$ and $g(x) = \Sigma_{i=2}^{\infty}(4/3^{i+1})x^i$.
   (a) Does $f(x) - g(x)$ converge? Explain.
   (b) If it converges, write an expression for the difference and find its domain.
2. True or false:
   (a) The derivative of $\Sigma_{i=0}^{\infty}(x^i/i!)$ is $\Sigma_{i=0}^{\infty}(ix^{i-1}/i!) = \Sigma_{i=0}^{\infty}[x^{i-1}/(i-1)!]$.
   (b) The integral of $\Sigma_{i=0}^{\infty}(x^i/i!) = \Sigma_{i=0}^{\infty}[x^{i+1}/(i+1)i!] = \Sigma_{i=0}^{\infty}[x^{i+1}/(i+1)!]$.
3. Find all x for which the following power series converge:
   (a) $\Sigma_{i=1}^{\infty}[(x-3)^i/i]$
   (b) $\Sigma_{n=1}^{\infty}(1+1/n)^{(n^2)}x^n$
   (c) $\Sigma_{n=1}^{\infty}[(x+2)^n \cos \pi n/4^n]$
4. The ancient Greeks believed that an average centaur's speed was given by $f(t) = \Sigma_{n=1}^{\infty}[2n^2/(n^3+n)]t^n$, where t is a unit of time.
   (a) For what t does $f(t)$ converge?
   (b) Within the radius of convergence, find a formula for the distance travelled by the centaur.
   (c) Find a formula for the centaur's acceleration and determine the radius of convergence for the acceleration.

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ANSWERS TO PREREQUISITE QUIZ

1. (a) $5x^4 - 6x + 1$

   (b) $12x + 4x^3$

2. (a) $-3/4$

   (b) $t^6/6 + t^3/3 + 3t + C$

3. (a) $\lim_{i\to\infty} |(3/2i)^i|^{1/i} = \lim_{i\to\infty}(3/2i) = 0 < 1$, so $\Sigma_{n=1}^{\infty}(3/2n)^n$ converges by the root test.

   (b) $\lim_{i\to\infty} |(100i/i!)/[100(i-1)/(i-1)!]| = \lim_{i\to\infty}[1/(i-1)] = 0$, so $\Sigma_{n=1}^{\infty}(100n/n!)$ converges by the ratio test.

ANSWERS TO SECTION QUIZ

1. (a) $f(x) - g(x)$ converges if $f(x)$ and $g(x)$ both converge. In this case, $f(x)$ and $g(x)$ both converge if $-3 < x < 3$.

   (b) $f(x) - g(x) = 2 + 2x/3 + \Sigma_{i=2}^{\infty}(2/3^i - 4/3^{i+1})x^i = 2 + 2x/3 - \Sigma_{i=2}^{\infty}(2/3^{i+1})x^i$. Its radius of convergence is $-3 < x < 3$.

2. (a) False; the index for the derivative should go from $i = 1$ to ∞ .

   (b) True

3. (a) $2 \leq x \leq 4$

   (b) $-1/e < x < 1/e$

   (c) $-6 < x < 2$

4. (a) $-1 \leq t < 1$

   (b) Distance $= \int f(t)dt = \Sigma_{n=1}^{\infty}[2n^2/(n^3 + n)][t^{n+1}/(n + 1)]$

   (c) Acceleration $= f'(t) = \Sigma_{n=1}^{\infty}[2n^2/(n^3 + n)]nt^{n-1}$; radius of convergence $= 1$.

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12.5 Taylor's Formula

PREREQUISITES

1. Recall how to differentiate and integrate power series (Section 12.4).
2. Recall how to calculate the radius of convergence of a power series (Section 12.4).
3. Recall how to use the linear approximation (Section 1.6).

PREREQUISITE QUIZ

1. Find all x for which the power series $\Sigma_{i=1}^{\infty}[(-1)^i(x-2)^i/i]$ converges.
2. Let $f(x) = \Sigma_{n=0}^{\infty}[(x-3)^n/n!]$.
   (a) Compute $\int f(x)dx$.
   (b) Compute $(d/dx)f(x)$.
3. Use the linear approximation to estimate $(2.11)^3$.

GOALS

1. Be able to find the Taylor series for a given function.
2. Be able to estimate the error in using the Taylor approximation.
3. Be able to state the Taylor series expansions of commonly used functions.

STUDY HINTS

1. Definitions. $\Sigma_{i=0}^{\infty}[f^{(i)}(x_0)/i!](x-x_0)^i$ is called the Taylor series of f about the point $x = x_0$. $f^{(i)}(x_0)$ is the i^{th} derivative of f computed at x_0 . If $x_0 = 0$, the series is known as a Maclaurin series.

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2. Error estimates. There are two forms of the remainder for Taylor's series. The integral form is $\int_{x_0}^{x} [(x - t)^n/n!] f^{(n+1)}(t)dt$. The derivative form is $[f^{(n+1)}(c)/(n + 1)!](x - x_0)^{n+1}$, where c is some number between x and x_0 . You estimate the remainder by inserting an inequality for f^{n+1} .

3. Proving convergence. Study Examples 4(a) and 4(b) well. They demonstrate how convergence is often proven for Taylor's series.

4. Taylor series limitations. The paragraph following Example 4 describes interesting limitations of Taylor's series. The first problem is that we may be interested in a point outside the radius of convergence. The second problem is that all of the derivatives may be zero for the Maclaurin series, yet the function isn't zero.

5. Usefulness of Taylor's series. You may be wondering why you should bother with learning about Taylor's series. It can be used to approximate functions whose values could only be found in tables before the invention of calculators. The Taylor approximation improves the linear one.

6. Important series. The box on p. 600 lists the most important Taylor series that you will encounter. Although you should be able to derive each one, it is recommended that you memorize the series in order to save time, except perhaps the series for $\ln(1 + x)$ which can be rapidly derived by integrating $1/(1 + x) = 1 - x + x^2 - x^3 + \ldots$ [$1/(1 + x)$ is a geometric series whose ratio is $-x$.] .

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. Since $\sin x = \Sigma_{i=0}^{\infty}[(-1)^i x^{2i+1}/(2i + 1)!]$, we replace x with $3x$ to get $\sin 3x = \Sigma_{i=0}^{\infty}[(-1)^i (3x)^{2i+1}/(2i + 1)!] = 3x - 9x^3/2 + 81x^5/80 - 243x^7/1120 + \ldots$.

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5. Use the method of Example 2. $f(x) = (1 + x^2 + x^4)^{-1}$, $f'(x) = -1(1 + x^2 + x^4)^{-2}(2x + 4x^3)$, $f''(x) = 2(1 + x^2 + x^4)^{-3}(2x + 4x^3)^2 - (1 + x^2 + x^4)^{-2}(2 + 12x^2)$, and $f'''(x) = -6(1 + x^2 + x^4)^{-4}(2x + 4x^3)^3 + 4(1 + x^2 + x^4)^{-3}(2x + 4x^3)(2 + 12x^2) + 2(1 + x^2 + x^4)^{-3}(2x + 4x^3)(2 + 12x^2) - (1 + x^2 + x^4)^{-2}(24x)$. Evaluating at $x_0 = 1$, we have $f(1) = 1/3$, $f'(1) = -2/3$, $f''(1) = 2(36)/27 - 14/9 = 10/9$, and $f'''(1) = -6(216)/81 + 4(6)(14)/27 + 2(6)(14)/27 - 24/9 = -16 + 56/3 - 24/9 = 0$. Thus, $1/(1 + x^2 + x^4) \approx 1/3 + (-2/3)(x - 1) + (10/9)(x - 1)^2/2 + 0(x - 1)^3 = 1/3 - 2(x - 1)/3 + 5(x - 1)^2/9 + 0(x - 1)^3$.

9. (a) Substituting $-x^2 - x^4$ for x in the formula for $1/(1 - x)$ yields $1 + (-x^2 - x^4) + (-x^2 - x^4)^2 + (-x^2 - x^4)^3 + \ldots = 1 - x^2 + x^6 + \ldots$.

   (b) By using the formula for the Maclaurin series, we know that $1 = f^{(5)}(0)/6!$; therefore, $f^{(5)}(0) = 6! = 720$.

13. $f(x) = x^{1/2}$, so $f'(x) = (1/2)x^{-1/2}$, $f''(x) = (1/2)(-1/2)x^{-3/2}$, $f'''(x) = (1/2)(-1/2)(-3/2)x^{-5/2}$, $\ldots$, $f^{(n)}(x) = (-1)^{n+1}[(2n - 3)!/2(n - 2)!]x^{-(2n-1)/2}/2^n$. Thus, f is infinitely differentiable. At $x_0 = 1$, $f(1) = 1$, $f'(1) = 1/2$, $f''(1) = -1/4$, and $f'''(1) = 3/8$. Thus, $\sqrt{x} = 1 + (1/2)(x - 1) - (1/8)(x - 1)^2 + (1/16)(x - 1)^3 - \ldots$. We need to show that $|[(2n - 3)!/2(n - 2)!]x^{-(2n-1)/2}/2^n| \leq CM^n$. Since $(2n - 3)! \leq (2n)^n$, choose $M = (2n)/2 = n$ and C to be the maximum of $x^{-(2n-1)/2}$ on I .

   Alternatively, we can assume that x is in the interval $(0,2)$ and use the binomial series $(1 + u)^\alpha = \Sigma_{i=0}^{\infty}[\alpha(\alpha - 1)\cdots(\alpha - i + 1)u^i/i!]$. Then let $\alpha = 1/2$ and $u = x - 1$ to get $(1 + x - 1)^{1/2} = x^{1/2} = 1 + (x - 1)/2 + [(1/2)(-1/2)/2](x - 1)^2 + [(1/2)(-1/2)(-3/2)/3!](x - 1)^3 + \ldots = 1 + (x - 1)/2 - (x - 1)^2/8 + (x - 1)^3/16 - \ldots$.

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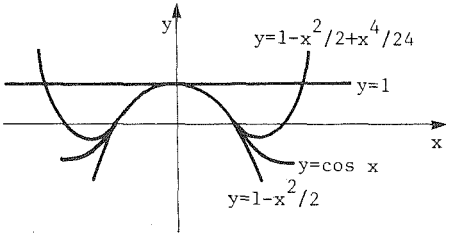
17. 

The Maclaurin series for $\cos x$ is $1 - x^2/2 + x^4/4! - \ldots$. Thus, $f_0(x) = f_1(x) = 1$, $f_2(x) = f_3(x) = 1 - x^2/2$, and $f_4(x) = 1 - x^2/2 + x^4/24$.

21. Since $2.5 = (3/2)\cdot(5/3)$, we have $\ln(2.5) = \ln(3/2) + \ln(5/3) = \Sigma_{i=1}^{\infty}[(-1)^{i+1}(1/2)^i/i] + \Sigma_{i=1}^{\infty}[(-1)^{i+1}(2/3)^i/i]$. The error is $R_n(x) = [f^{(n+1)}(c)/(n+1)!](x - x_0)^{n+1}$, where $f(x) = \ln(1 + x)$ and $x_0 = 0$. We have $f'(x) = (1 + x)^{-1}$, $f''(x) = -(1 + x)^{-2}$, $f'''(x) = 2(1 + x)^{-3}$, $\ldots$, $f^{(n)}(x) = (-1)^{n+1}(n-1)!(1 + x)^{-n}$. On $[0,2/3]$, $R_4 < 0.007$ and on $[0,1/2]$, $R_4 < 0.002$, so for $n = 4$, the total error is less than 0.009 . Therefore, $\ln(2.5) \approx (1/2 - 1/8 + 1/24 - 1/64) + (2/3 - 2/9 + 8/81 - 4/81) \approx 0.9$.

25. (a) $f(x) \approx f(x_0) + [f'(x_0)](x - x_0) + [f''(x_0)](x - x_0)^2/2$.
$\int_{x_0-R}^{x_0+R} f(x)dx = \int_{x_0-R}^{x_0+R}\{f(x_0) + [f'(x_0)](x - x_0) + [f''(x_0)](x - x_0)^2/2\}dx = \{[f(x_0)](x - x_0) + [f'(x_0)](x - x_0)^2/2 + [f''(x_0)](x - x_0)^3/6\}\Big|_{x_0-R}^{x_0+R} = f(x_0)R + f'(x_0)R^2/2 + f''(x_0)R^3/6 - [f(x_0)(-R) + f'(x_0)R^2/2 + f''(x_0)(-R)^3/6] = 2Rf(x_0) + 2f''(x_0)R^3/3!$. By equation (4) in the text, the remainder is $\left|\int_{x_0-R}^{x_0+R}[(x - t)^3 f'''(t)/3!]dt\right| \le \left|\int_{x_0}^{x_0+R}\{[(x_0 + R) - t]^3 f'''(t)/3!\}dt\right| + \left|\int_{x_0}^{x_0-R}\{[(x_0 - R) - t]^3 f'''(t)/3!\}dt\right| \le \left|\int_{x_0}^{x_0+R}\{[(x_0 + R) - t]^3 M_3/3!\}dt\right| + \left|\int_{x_0}^{x_0-R}\{[(x_0 - R) - t]^3 M_3/3!\}dt\right| = \left|(x_0 + R - t)^4 M_3/4!\right|_{x_0}^{x_0+R}\Big| + \left|(x_0 - R - t)^4 M_3/4!\right|_{x_0}^{x_0-R}\Big| = R^4 M_3/4! + R^4 M_3/4! = R^4 M_3/12$, where M_3 is the maximum value $|f'''(x)|$ takes on for x in $[x_0 - R, x_0 + R]$.

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25. (b) Let $x_0 = 0$, $R = 1/2$, and $f(x) = 1/\sqrt{1 + x^2}$. Then $f'(x) = -x(1 + x^2)^{-3/2}$ and $f''(x) = -(1 + x^2)^{-3/2} - 3x^2(1 + x^2)^{-5/2}$. $f(0) = 1$ and $f''(0) = -1$. Therefore, $\int_{-1/2}^{1/2}\left[dx/\sqrt{1 + x^2}\right] \approx 2(1/2)1 + 2(-1)(1/2)^3/6 = 1 - 1/24 = 23/24 \approx 0.958$. Simpson's rule for $n = 4$, gives $(1/12)[f(-1/2) + 4f(-1/4) + 2f(0) + 4f(1/4) + f(1/2)] = (1/12)[4/\sqrt{5} + 32/\sqrt{17} + 1] = 0.879$.

29. $\lim_{x\to 0}(1/x \sin x - 1/x^2) = \lim_{x\to 0}[(x - \sin x)/x^2 \sin x]$. Now, $-\sin x = -x + x^3/3! - x^5/5! + x^7/7! - \ldots$ and $x - \sin x = x^3/3! - x^5/5! + x^7/7! - \ldots$. The denominator is $x^3 - x^5/3! + x^7/5! - \ldots$, so $(x - \sin x)/x^2 \sin x = (x^3/3! - x^5/5! + x^7/7! - \ldots)/(x^3 - x^5/3! + x^7/5! - \ldots) = (1/6 - x^2/5! + \ldots)/(1 - x^2/3! + \ldots)$ by dividing by x^3/x^3. Since the term involving x^n tends to 0 as x approaches 0, we get the limit as $1/6$.

33. By the sum rule, $1/(1 - x) - 1/(1 + x) = \Sigma_{n=0}^{\infty} x^n - \Sigma_{n=0}^{\infty}(-1)^n x^n = (1 + x + x^2 + x^3 + \ldots) - (1 - x + x^2 - x^3 + \ldots) = 2x + 2x^3 + \ldots = \Sigma_{n=0}^{\infty} 2x^{2n+1}$ for $|x| < 1$.

37. (a) Multiplying out, $f(x) = (1 + x^2)^2 = 1 + 2x^2 + x^4$.

(b) Since $f(x) = (1 + x^2)^2$, $f'(x) = 2(1 + x^2)\cdot 2x$, $f''(x) = 4(1 + x^2) + 8x^2$, $f'''(x) = 24x$, $f^{(4)}(x) = 24$, and $f^{(k)}(x) = 0$ for all $k > 4$. Thus $f(0) = 1$, $f'(0) = 0$, $f''(0) = 4$, $f'''(0) = 0$, $f^{(4)}(0) = 24$, and $f^{(k)}(0) = 0$ for all $k > 4$. Hence, the Maclaurin series for $f(x) = (1 + x^2)^2$ is $f(x) = f(0) + f'(0)(x - 0) + f''(0)(x - 0)^2/2! + f'''(0)x^3/3! + f^{(4)}(0)x^4/4! + \ldots = 1 + 0 + (4/2)x^2 + 0 + (24/4!)x^4 + 0 + 0 + \ldots = 1 + 2x^2 + x^4$.

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41. Since $f(x) = \sec x$, $f(0) = 1$. $f'(x) = \sec x \tan x$, so $f'(0) = 0$. $f''(x) = \sec x \tan^2 x + \sec^3 x$, so $f''(0) = 1$, $f'''(x) = \sec x \tan^3 x + 2 \sec^3 x \tan x + 3 \sec^3 x \tan x$, so $f'''(0) = 0$. Thus, $a_0 = f(0) = 1$; $a_1 = f'(0) = 0$; $a_2 = f''(0)/2 = 1/2$; $a_3 = f'''(0)/3! = 0$.

45. We know that $\cos x = \sum_{i=0}^{\infty} [(-1)^i x^{2i}/(2i)!]$. Therefore, $1 - \cos x = \sum_{i=1}^{\infty} [(-1)^{i+1} x^{2i}/(2i)!]$. Thus, $(1 - \cos x)/x^2 = \sum_{i=1}^{\infty} [(-1)^{i+1} x^{2i-2}/(2i)!] = 1/2 - x^2/4! + x^4/6! - \ldots$.

49. Let $f(x) = \ln x$. Then, $f'(x) = x^{-1}$; $f''(x) = -x^{-2}$; $f'''(x) = 2x^{-3}$; $f''''(x) = -6x^{-4}$. The Taylor expansion of degree 4 for $\ln x$ is $\ln x_0 + x_0^{-1}(x - x_0) - x_0^{-2}(x - x_0)^2/2! + 2x_0^{-3}(x - x_0)^3/3! - 6x_0^{-4}(x - x_0)^4/4! = \ln x_0 + (x - x_0)/x_0 - (x - x_0)^2/2x_0^2 + (x - x_0)^3/3x_0^3 - (x - x_0)^4/4x_0^4$.

(a) For $x_0 = 1$, the polynomial is $(x - 1) - (x - 1)^2/2 + (x - 1)^3/3 - (x - 1)^4/4$.

(b) For $x_0 = e$, the polynomial is $1 + (x - e)/e - (x - e)^2/2e^2 + (x - e)^3/3e^3 - (x - e)^4/4e^4$.

(c) For $x_0 = 2$, the polynomial is $\ln 2 + (x - 2)/2 - (x - 2)^2/8 + (x - 2)^3/24 - (x - 2)^4/64$.

53. Since $f(x) = \sin e^x$, $f(0) = \sin 1$; $f'(x) = e^x \cos e^x$, $f'(0) = \cos 1$; $f''(x) = e^x \cos e^x - e^{2x} \sin e^x$, $f''(0) = \cos 1 - \sin 1$; $f'''(x) = e^x \cos e^x - e^{2x} \sin e^x - 2e^{2x} \sin e^x - e^{3x} \cos e^x$, $f'''(0) = -3 \sin 1$. The first four terms are $\sin 1 + (\cos 1)x + [(\cos 1 - \sin 1)/2]x^2 - [(\sin 1)/2]x^3$.

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57. (a) $f'(0) = \lim_{\Delta x\to 0}\{[f(\Delta x) - f(0)]/\Delta x = \lim_{\Delta x\to 0}\{[(\sin \Delta x)/\Delta x - 1]/\Delta x\} = \lim_{\Delta x\to 0}[(\sin \Delta x - \Delta x)/(\Delta x)^2] = \lim_{\Delta x\to 0}[(\cos \Delta x - 1)/2(\Delta x)] = \lim_{\Delta x\to 0}[-\sin \Delta x/2] = 0$ (l'Hôpital's rule). For $x \neq 0$, $f'(x) = (x \cos x - \sin x)/x^2$. Therefore, $f''(0) = \lim_{\Delta x\to 0}\{[f'(\Delta x) - f'(0)]/\Delta x\} = \lim_{\Delta x\to 0}\{[(\Delta x \cos \Delta x - \sin \Delta x)/(\Delta x)^2]/\Delta x\} = \lim_{\Delta x\to 0}[(\Delta x \cos \Delta x - \sin \Delta x)/(\Delta x)^3] = \lim_{\Delta x\to 0}[(\cos \Delta x - \Delta x \sin \Delta x - \cos \Delta x)/3(\Delta x)^2] = \lim_{\Delta x\to 0}[(-\sin \Delta x - \Delta x \cos \Delta x)/6(\Delta x)] = \lim_{\Delta x\to 0}[(-\cos \Delta x - \cos \Delta x + \Delta x \sin \Delta x)/6] = -1/3$ (l'Hôpital's rule). For $x \neq 0$, $f''(x) = [x^2(\cos x - x \sin x - \cos x) - (x \cos x - \sin x)2x]/x^4 = (-x^2 \sin x - 2x \cos x + 2 \sin x)/x^3$. Therefore, $f'''(0) = \lim_{\Delta x\to 0}\{[f''(\Delta x) - f''(0)]/\Delta x\} = \lim_{\Delta x\to 0}\{[-(\Delta x)^2 \sin \Delta x - 2\Delta x \cos \Delta x + 2 \sin \Delta x + (\Delta x)^3/3]/(\Delta x)^4\} = \lim_{\Delta x\to 0}\{[-2\Delta x \sin \Delta x - (\Delta x)^2 \cos \Delta x - 2 \cos \Delta x + 2\Delta x \sin \Delta x + 2 \cos \Delta x + (\Delta x)^2]/4(\Delta x)^3\} = \lim_{\Delta x\to 0}\{[-2\Delta x \cos \Delta x + (\Delta x)^2 \sin \Delta x + 2\Delta x]/12(\Delta x)^2\} = \lim_{\Delta x\to 0}\{[-2 \cos \Delta x + 2\Delta x \sin \Delta x + 2\Delta x \sin \Delta x + (\Delta x)^2 \cos \Delta x + 2]/24\Delta x\} = \lim_{\Delta x\to 0}\{[-2 \sin \Delta x + 4 \sin \Delta x + 4 \Delta x \cos \Delta x + 2\Delta x \cos \Delta x - (\Delta x)^2 \sin \Delta x]/24\} = 0$.

(b) $\sin x = \Sigma_{i=0}^{\infty}[(-1)^i x^{2i+1}/(2i + 1)!]$, so $\sin x/x = \Sigma_{i=0}^{\infty}[(-1)^i x^{2i}/(2i + 1)!] = 1 - x^2/3! + x^4/5! - x^6/7! + \ldots$.

SECTION QUIZ

1. (a) What is the value of $\Sigma_{i=0}^{\infty}[(-1)^i/i!]$?

   (b) What is the value of $\Sigma_{i=0}^{\infty}[(-1)^i \pi^{2i+1}/(2i)!]$?

2. Use a Taylor series to prove that $\lim_{x\to 0}(\sin x/x) = 1$.

3. Find the Taylor series expansion of $f(x) = x^3 + 2x - 2$ about $x = 2$ and simplify your answer.

4. Find the sum of $\Sigma_{n=0}^{\infty}[(-1)^n/4^n(2n + 1)]$. (Hint: If $f(x) = \tan^{-1}(x/2)$, what is $f'(x)$?)

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5. Find the Taylor series expansion of the following functions around the indicated point:

   (a) $y = \sin^{-1} x$ about $x = 0$

   (b) $y = 1/\sqrt[3]{x - 2}$ about $x = 3$

6. A frustrated calculus student was overheard saying, "Taylor's formula is useless." That evening, the student was visited by Taylor's ghost. The ghost explained that he travels throughout the night with speed $f(t) = t \cos t$ at time t to haunt nonbelievers of Taylor's formula. He will return to haunt calculus students who can't find a fourth-order approximation to his speed at $t = 0.2$. He will return tomorrow for the answer and an upper bound for the error. Find the answers which will prevent return visits by Taylor's ghost.

ANSWERS TO PREREQUISITE QUIZ

1. $1 < x \leq 3$

2. (a) $\sum_{n=0}^{\infty} [(x - 3)^{n+1}/(n + 1)!]$

   (b) $\sum_{n=1}^{\infty} [n(x - 3)^{n-1}/n!]$

3. 9.32

ANSWERS TO SECTION QUIZ

1. (a) $1/e$

   (b) $-\pi$

2. The Taylor expansion of $\sin x$ about $x = 0$ is $x - x^3/3! + x^5/5! - \ldots$, so $\sin x/x = 1 - x^2/3! + x^4/5! - \ldots$, and the limit is 1.

3. $f(x) = 10 + 14(x - 2) + 6(x - 2)^2 + (x - 2)^3 = x^3 + 2x - 2$.

4. $\tan^{-1}(1/2)$

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5. (a) $x + x^3/6 + \sum_{n=2}^{\infty} [(2n - 1)!/2^{n+1} n!(n - 1)!(2n + 1)] x^{2n+1}$

(b) $\sum_{n=0}^{\infty} [(-1)^n (2n + 1)!(1)(4)(7) \ldots (2n + 1)/3^n n!] (x - 3)^n$

6. Expand around $t = 0$; $t \cos t \approx t - t^3/2$, which is 0.196 at $t = 0.2$. Error $\leq |- 3 \cos(0.2) - (0.2) \sin (0.2)|/120(5)^5 < 1/120(5)^4$.

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12.6 Complex Numbers

PREREQUISITES

1. Recall the power series expansions for $1/(1 + x)$, $\cos x$, $\sin x$, and e^x (Section 12.5).
2. Recall how to use polar coordinates (Section 5.1).

PREREQUISITE QUIZ

1. Plot the polar coordinates $(2,-5\pi/8)$ in the xy-plane.
2. Convert the polar coordinates $(-1, 3\pi/4)$ into cartesian coordinates.
3. Convert the cartesian coordinates $(2,-2)$ into polar coordinates.
4. Write down the power series expansions for the following:
   (a) $\cos 2x$
   (b) e^{-x}

GOALS

1. Be able to explain the relationship between trigonometry and the imaginary numbers.
2. Be able to do basic algebra with complex numbers.
3. Be able to write down the polar representation of a complex number and use this form to extract roots.

STUDY HINTS

1. Imaginary numbers defined. The basic definition is $\sqrt{-1} = i$. You should know that $e^{ix} = \cosh ix + \sinh ix = \cos x + i \sin x$. Therefore, $\cosh ix = \cos x$ and $\sinh ix = i \sin x$. In addition, by letting $x = \pi$, we get $e^{i\pi} = -1$.

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2. Algebra of complex numbers. When you add and mulitply, i acts just like an ordinary number except that you must remember that $i^2 = -1$.

3. Terminology. As with any new subject, an essential starting point is learning the vocabulary. Therefore, learning the vocabulary presented on p. 611 will aid you in understanding the discussions about complex numbers.

4. Notation. Many times, (a,b) will denote a complex number. It is understood to stand for $z = a + bi$.

5. Removing i from denominator. To standardize answers, it is desirable to remove i from the denominator just as we desired to have denominators without radicals. If $a + bi$ occurs in the denominator, multiplying by $(a - bi)/(a - bi)$ yields $a^2 + b^2$ in the denominator and removes the symbol i from the denominator.

6. Properties of complex numbers. The properties presented on p. 612 are helpful in manipulations. Note their similarities to properties of real numbers.

7. Exponential properties. Multiplication and division of exponents are done exactly as they are done with real numbers.

8. Polar representation. $z = re^{i\theta}$ is called the polar representation of $z = a + ib$. (r,θ) are the polar coordinates of (a,b). Notice how easy it is to mulitply two imaginary numbers which are written in polar representation.

9. DeMoivre's formula. The n^{th} roots of an imaginary number $\rho e^{i\phi}$ are located on a circle centered at the origin with radius $\sqrt[n]{\rho}$. One root is located at an angle of ϕ/n from the positive real axis. There are $n - 1$ other roots equally spaced on the circle, separated by angles of $2\pi/n$. The validity of DeMoivre's formula is easily demonstrated using the polar representation to raise the claimed roots to the n^{th} power.

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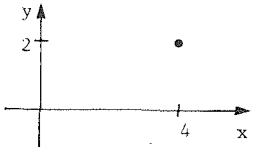
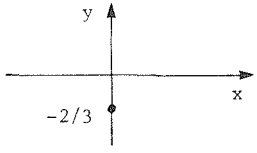
SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. Using $e^{ix} = \cos x + i \sin x$, we get $e^{-\pi i/2} = \cos(-\pi/2) + i \sin(-\pi/2) = -i$.

5. 

The complex number $x + yi$ is plotted at the point (x,y) in the xy-plane. Thus, $4 + 2i$ is plotted at $(4,2)$.

9. 

The complex number $x + yi$ is plotted at the point (x,y) in the xy-plane. Thus, $(-2/3)i$ is plotted at $(0,-2/3)$.

13. We need to add the real and imaginary parts separately. $(1 + 2i) - 3(5 - 2i) = (1 + 2i) + (-15 + 6i) = -14 + 8i$.

17. Mulitply by $(5 + 3i)/(5 + 3i)$ to get $(5 + 3i)/(25 - 9i^2) = (5 + 3i)/34$.

21. If $z^2 + 3 = 0$, then $z^2 = -3$ or $z = \pm\sqrt{-3} = \pm\sqrt{3}i$.

25. By the quadratic formula, $z = (7 \pm \sqrt{49 + 4})/2 = (7 \pm \sqrt{53})/2$.

29. From Example 4(d) , $\sqrt{i} = \pm(\sqrt{2}/2)(1 + i)$, so $\sqrt{-16i} = \sqrt{16}\sqrt{i}\sqrt{-1} = 4i\sqrt{i} = \pm 4i(\sqrt{2}/2)(1 + i) = \pm(\sqrt{2}/2)(4i - 4) = \pm(2\sqrt{2}i - 2\sqrt{2})$.

33. Since $(1 + 2i)^2 = (-3 + 4i)$, $(10 + 5i)/(1 + 2i)^2 = (10 + 5i)/(-3 + 4i)$. Multiply by $(-3 - 4i)/(-3 - 4i)$ to get $(-30 - 55i - 20i^2)/(9 - 16i^2) = (-10 - 55i)/25$, so the imaginary part is $-55/25 = -11/5$.

37. The complex conjugate of $a + bi$ is $a - bi$, so $\bar{z} = 5 - 2i$.

41. Multiply by i/i to get $(2 - i)/3i = (2i + 1)/(-3)$, so $\bar{z} = -1/3 + 2i/3$.

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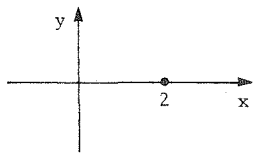
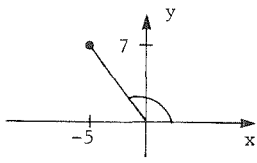
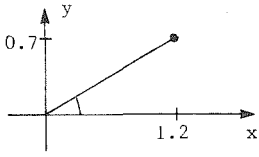
45. Here, $z = 3 + 0i$, so the complex conjugate is $\bar{z} = 3 - 0i = 3$.

49. 

$z = 2$ is plotted at the left. $|z|$ is the distance from the origin, which is 2 in this case. θ is the angle from the positive x-axis, which is 0 in this case.

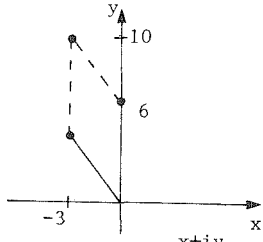
53. 

$z = -5 + 7i$ is plotted at the left. $|z| = \sqrt{a^2 + b^2} = \sqrt{25 + 49} = \sqrt{74}$. The argument, θ, is the angle from the positive x-axis. In this case, $\theta = \cos^{-1}(-5/\sqrt{74}) \approx 2.19$.

57. 

$z = 1.2 + 0.7i$ is plotted at the left. Here, $|z| = \sqrt{a^2 + b^2} = \sqrt{1.44 + 0.49} = \sqrt{1.93}$. The argument is $\theta = \cos^{-1}(1.2/\sqrt{1.93}) \approx 0.53$.

61. From Exmaple 7(b), we know that $\overline{z^n} = \bar{z}^n$, so $\overline{(8 - 3i)^4} = \overline{(8 - 3i)}^4 = (8 + 3i)^4$.

65. 

Two sides of a parallelogram are determined by line segments joinging the origin to z_1 and z_2. The sum is the fourth vertex of the parallelogram. In this case, the sum is $-3 + 10i$.

69. We know that $e^{x+iy} = e^x \cdot e^{iy} = e^x(\cos y + i \sin y)$. Thus, $|e^z| = (e^{2x}\cos^2 y + e^{2x}\sin^2 y)^{1/2} = [e^{2x}(\cos^2 y + \sin^2 y)]^{1/2} = e^x$. The argument is $\theta = \cos^{-1}(a/r)$, where $z = a + bi$ and $r = |z|$. Therefore, $\theta = \cos^{-1}(e^x \cos y / e^x) = \cos^{-1}(\cos y) = y$.

73. Use the fact that $e^{ix} = \cos x + i \sin x$. Thus, $e^{1-\pi i/2} = e \cdot e^{-\pi i/2} = e[\cos(-\pi/2) + i \sin(-\pi/2)] = e(-i) = -ei$.

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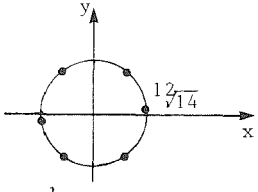
77. We have $f(z) = 1/(2 + i)^2 = 1/(3 + 4i)$. Multiply by $(3 - 4i)/(3 - 4i)$ to get $(3 - 4i)/(9 - 16i^2) = (3 - 4i)/25$.

81. By the law of exponents, $e^{i(\theta+3\pi/2)} = e^{i\theta} \cdot e^{3\pi i/2}$. Using $e^{ix} = \cos x + i \sin x$, we get $e^{i\theta} \cdot e^{3\pi i/2} = (\cos\theta + i \sin\theta)(\cos(3\pi/2) + i \sin(3\pi/2)) = (\cos\theta + i \sin\theta)(-i) = \sin\theta - i\cos\theta$.

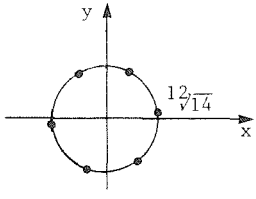
85. If $z = a + ib$, then $r = \sqrt{a^2 + b^2}$ and $\theta = \cos^{-1}(a/r) = \sin^{-1}(b/r)$. Here, $z = 1 + i$, so $r = \sqrt{1 + 1} = \sqrt{2}$ and $\theta = \cos^{-1}(1/\sqrt{2}) = \pi/4$. Thus, $1 + i = \sqrt{2}e^{i\pi/4}$.

89. If $z = a + ib$, then $r = \sqrt{a^2 + b^2}$ and $\theta = \cos^{-1}(a/r) = \sin^{-1}(b/r)$. Here, $z = 7 - 3i$, so $r = \sqrt{49+9} = \sqrt{58}$ and $\theta = \cos^{-1}(7/\sqrt{58}) = \sin^{-1}(-3/\sqrt{58}) \approx -0.40$. Thus, $7 - 3i = \sqrt{58}\, e^{i(-0.40)}$.

93. Squaring gives $(3 + 4i)^2 = 9 + 24i + 16i^2 = -7 + 24i$. So $r = \sqrt{7^2 + 24^2} = 25$ and $\theta = \cos^{-1}(-7/25) \approx 1.85$. Thus, $(3 + 4i)^2 = 25e^{i(1.85)}$.

97. 

Write the number as $\rho e^{i\phi}$. Then the n^{th} roots of $\rho e^{i\phi}$ are given by $\sqrt[n]{\rho}\, e^{i(\phi/n+2\pi k/n)}$, where $k = 0, 1, 2, \ldots, n - 1$. For $\sqrt{5} + 3i$, $\rho = \sqrt{5 + 9} = \sqrt{14}$ and $\phi = \cos^{-1}(\sqrt{5}/14) = \sin^{-1}(3/\sqrt{14}) \approx 0.93$. Thus, $\sqrt{5} + 3i = \sqrt{14}\, e^{i(0.93)}$ and the sixth roots are $\sqrt[12]{14}\, e^{i(0.155+2\pi k/6)}$, where $k = 0, 1, 2, 3, 4,$ and 5.



For $3 + \sqrt{5}i$, $\rho = \sqrt{9 + 5} = \sqrt{14}$ and $\phi = \cos^{-1}(3/\sqrt{14}) = \sin^{-1}(\sqrt{5}/14) \approx 0.64$. Thus, $3 + \sqrt{5}\, i = \sqrt{14}\, e^{i(0.64)}$ and the sixth roots are $\sqrt[12]{14}\, e^{i(0.107+ 2\pi k/6)}$, where $k = 0, 1, 2, 3, 4,$ and 5 .

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101. For $z = 1$, $w = e^0$, so the sixth roots are $e^{\pi ik/3}$. Thus, the roots are located on the unit circle at angles of 0, $\pi/3$, $2\pi/3$, π, $4\pi/3$, and $5\pi/3$. The tenth roots of e^0 are $e^{\pi ik/5}$, which are located on the unit circle at angles of 0, $\pi/5$, $2\pi/5$, $3\pi/5$, $4\pi/5$, π, $6\pi/5$, $7\pi/5$, $8\pi/5$, and $9\pi/5$. The sixth and tenth roots of 1 share common points at 0 and π, which correspond to $z = \pm 1$.

105. The quadratic formula gives us the roots $z = (-2 \pm \sqrt{4 - 4i})/2 = -1 \pm \sqrt{1 - i}$. Now, if $\sqrt{1 - i} = a + bi$, then $(a + bi)^2 = 1 - i$ or $a^2 - b^2 = 1$ and $2ab = -1$. Since $b = -1/2a$, $a^2 - b^2 = 1$ becomes $a^2 - 1/4a^2 = 1$ or $4a^4 - 4a^2 - 1 = 0$. Therefore, $a^2 = (4 \pm \sqrt{32})/8 = (1 \pm \sqrt{2})/2$. Also, $b^2 = a^2 - 1 = (-1 \pm \sqrt{2})/2$. Since a and b are real numbers, we get $\sqrt{1 - i} = (1/\sqrt{2})[(1 + \sqrt{2})^{1/2} + (-1 + \sqrt{2})^{1/2}i]$ or $\sqrt{1 - i} = (1/\sqrt{2})[-(1 + \sqrt{2})^{1/2} - (-1 + \sqrt{2})^{1/2}i]$. Therefore, $z^2 + 2z + i = (1/\sqrt{2})[z - (1 + \sqrt{2})^{1/2} - (-1 + \sqrt{2})^{1/2}i] \times [z + (1 + \sqrt{2})^{1/2} + (-1 + \sqrt{2})^{1/2}i]$.

109. (a) Using the result of Example 2, we have $\cos i\theta = [e^{i(i\theta)} + e^{-i(i\theta)}]/2$ and $\sin i\theta = [e^{i(i\theta)} - e^{-i(i\theta)}]/2i$. Also, $\tan i\theta = \sin i\theta/\cos i\theta = (e^{-\theta} - e^{\theta})/(e^{-\theta} + e^{\theta})i$. We recognize this as $-\tanh \theta/i$. Thus, $\tan i\theta$ is the function $i \tanh \theta$.

(b) Using part (a), we have $r = \tanh \theta$ and $\phi = \pi/2$, so $\tan i\theta = (\tanh \theta)e^{i\pi/2}$.

113. (a) Note that $(z - 1)(z^{n-1} + z^{n-2} + \ldots + z + 1) = z^n - z^{n-1} + z^{n-1} - z^{n-2} + z^{n-2} - \ldots - z^2 + z^2 - z + z - 1 = z^n - 1$. Therefore, if $z^n - 1 = 0$, either $z - 1 = 0$ or $z^{n-1} + z^{n-2} + \ldots + z + 1 = 0$.

(b) If $z^{n-1} + z^{n-2} + \ldots + z + 1 = 0$, then $z(z^{n-1} + z^{n-2} + \ldots + z + 1) = z^n + (z^{n-1} + z^{n-2} + \ldots + z^2 + z) = 0$ or $z^n = -z^{n-1} - z^{n-2} - \ldots - z^2 - z = 1$, from the original equation.

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113. (c) The four roots of $z^4 = 1$ are 1, -1, i, $-i$. Therefore, the three roots of $z^3 + z^2 + z + 1 = 0$ are -1, i, $-i$, by part (b).

117. (a) If $z = x + iy$, then $e^z = e^x(\cos y + i \sin y) = -1$ when $x = 0$ and when $y = \pi + 2n\pi$, where n is an integer.

(b) You might define $\ln(-1) = i\pi$, though there are many values of z such that $e^z = -1$.

SECTION QUIZ

1. Let $u = i/(a + ib)$, where a and b are real numbers.
   (a) What is the imaginary part of u?
   (b) What is the real part of u?
   (c) What is $u \cdot \bar{u}$?
   (d) What is the polar representation of u?
2. What is e^{ix} in terms of sines and cosines?
3. Find all of the solutions of $z^4 + 1 = 0$.
4. Let $z_1 = 2 + 2i$ and $z_2 = 2\sqrt{3} - 2i$.
   (a) Convert z_1 and z_2 to their polar representation.
   (b) Compute $z_1 \cdot z_2$. Express your answer in both the $a + bi$ form and the polar representation.
   (c) Explain how multiplication of complex numbers can be related to the lengths and arguments of complex numbers.
5. Compute the following:
   (a) $(3 + 2i)^2$
   (b) $\bar{z}$ if $z = 1/(i - 2)$
   (c) $(5 + 3i)i - 2e$

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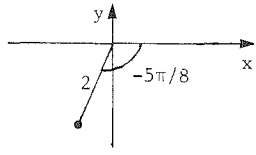
6. Ronnie was struggling through his calculus assignment when his fairy godfather appeared to offer his assistance. Asked what the problem was, Ronnie explained that he needed to do some root extractions. When the dumb fairy godfather heard this, he sent for the tooth fairy. Ronnie explained that he had to find the sixth roots of $2 + 3i$.

   (a) Fortunately, the tooth fairy understood Ronnie's problem. What answer did she give him? Express the answer in the form $a + bi$.

   (b) Plot all of the points representing $\sqrt[6]{2 + 3i}$ in the xy-plane.

   (c) Of all the answers in (a), find the complex conjugate of the one with the smallest positive argument.

ANSWERS TO PREREQUISITE QUIZ

1. 

2. $(\sqrt{2}/2, -\sqrt{2}/2)$

3. $(2\sqrt{2}, -\pi/4)$

4. (a) $1 - 4x^2/2! + 16x^4/4! - \ldots = \Sigma_{i=0}^{\infty}[(-1)^i(2x)^{2i}/(2i)!]$

   (b) $1 - x + x^2/2! - x^3/3! + \ldots = \Sigma_{i=0}^{\infty}[(-x)^i/i!]$

ANSWERS TO SECTION QUIZ

1. (a) $a/(a^2 + b^2)$

   (b) $b/(a^2 + b^2)$

   (c) $1/(a^2 + b^2)$

   (d) $re^{i\theta}$, where $r = 1/\sqrt{a^2 + b^2}$ and $\theta = \tan^{-1}(a/b)$

2. $e^{ix} = \cos x + i \sin x$

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3. $e^{i\pi/4}$, $e^{3\pi i/4}$, $e^{5\pi i/4}$, $e^{7\pi i/4}$

4. (a) $z_1 = 2\sqrt{2}\ \exp(\pi i/4)$ and $z_2 = 4\ \exp(-\pi i/6)$

   (b) $(4\sqrt{3} + 4) + (4\sqrt{3} - 4)i = 8\sqrt{2}\ \exp(\pi i/12)$

   (c) $(r_1\exp(i\theta_1))\ (r_2\exp(i\theta_2)) = r_1r_2\exp[(\theta_1 + \theta_2)i]$. The lengths are multiplied and the arguments are added.

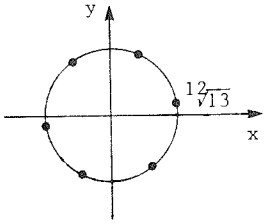
5. (a) $5 + 12i$

   (b) $(-2 + i)/5$

   (c) $-3 - 2e + 5i$

6. (a) $1.2 + 0.2i$; $0.4 + 1.2i$; $-0.8 + 1.0i$; $-1.2 - 0.2i$; $-0.4 - 1.2i$; $0.8 - 1.0i$

   (b)



   (c) $1.2 - 0.2i$

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12.7 Second-Order Linear Differential Equations

PREREQUISITES

1. Recall the quadratic formula (Section R.1).
2. Recall the polar representation for complex numbers (Section 12.6).
3. Recall basic differentiation and integration formulas (Section 7.1).
4. Recall how to solve the spring equation (Section 8.1).

PREREQUISITE QUIZ

1. What is the solution of $ax^2 + bx + c = 0$ if a, b, c are real and $a \neq 0$?
2. Let z be the complex number $5 + 5i$. What is the polar representation of z ?
3. In terms of trigonometric functions, what is e^{ix} ?
4. Solve $d^2y/dx^2 + 4y = 0$, $y'(0) = 2$, $y(0) = 3$.
5. Evaluate the following:
   (a) $\int e^{3t}dt$
   (b) $(d/dy)(\cos y + \sin y)$
   (c) $\int (x^2 + 2x)dx$

GOALS

1. Be able to solve differential equations of the form $ay'' + by' + cy = 0$.
2. Be able to find a solution for nonhomogeneous equations using the methods of variation of parameters or undetermined coefficients.

STUDY HINTS

1. Characteristic equation. The key to solving $ay'' + by' + cy = 0$ is finding the roots of the characteristic equation $ar^2 + br + c = 0$. This is easily solved using the quadratic formula.

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2. Distinct characteristic roots. If r_1 and r_2 are distinct characteristic roots, then the solution of $ay'' + by' + cy = 0$ is $y = c_1\exp(r_1x) + c_2\exp(r_2x)$; c_1 and c_2 are determined by the initial conditions. If r_1 and r_2 are complex, then the exponential part of the solution may be rewritten using $e^{ix} = \cos x + i \sin x$.

3. Repeated characteristic roots. If r_1 is the only characteristic root, i.e., $b^2 - 4ac = 0$, then the solution of $ay'' + by' + cy = 0$ is $(c_1 + c_2x)\exp(r_1x)$.

4. Method of reduction of order. When the roots repeat, simply look for a solution of the form $y = v \exp(r_1x)$, where v is a function of x .

5. Method of root splitting. This is an alternative derivation of the formula for repeated roots given in item 3 above (formula (5) on p. 619). You shouldn't worry if you don't understand it, as it won't be used later.

6. Damping. The equation $d^2x/dt^2 + \beta(dx/dt) + \omega^2x = 0$ describes damped harmonic motion. Whether we are dealing with an overdamped, critically damped, or underdamped case depends upon whether the characteristic equation has two, one, or no real roots. The solution can be written in terms of sine and cosine if no real roots exist and the solution is oscillatory. See Fig. 12.7.2.

7. Nonhomogeneous equations. $ay'' + by' + cy = F(x)$ is a nonhomogeneous equation. Notice that the right-hand side is a function of x only. It is a homogeneous equation if $F(x) = 0$. The general solution of the homogeneous equation (containing two arbitrary constants) is denoted by y_h . A specific solution of the nonhomogeneous equation (with no arbitrary constants) is called a particular solution and is denoted by y_p . Then $y_p + y_h$ is the general solution of the nonhomogeneous equation. Note

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7. (continued)

that y_p alone solves it and y_h adds zero to $F(x)$.

8. Method of undetermined coefficients. Depending on the form of the right-hand side, guess that the particular solution is a linear combination of sines, cosines, exponentials, and polynomials. Differentiate and substitute into the left-hand side. Then, solve for the constants. See Example 5. Make up an equation with $F(x)$ as a polynomial. Your guess for a particular solution should be a polynomial of the same degree.

9. Variation of parameters. If y_1 and y_2 are solutions of the homogeneous equation, we look for a solution of the form $y_p = v_1y_1 + v_2y_2$, where v_1 and v_2 are also functions of x. Differentiate and substitute into the original equation. We set $v_1'y_1 + v_2'y_2 = 0$ and then the equation becomes $v_1'y_1' + v_2'y_2' = F/a$. These can be solved simultaneously for v_1' and v_2' and integration of v_1' and v_2' yields a particular solution.

10. Damped forced oscillations. The box on p. 628 should not be memorized. If you encounter a forced oscillation question on an exam, look for a particular solution of the form $A\cos(\Omega t - \delta)$, where Ω is the forcing frequency, and add it to the general solution of the homogeneous equation.

11. Wronksians. If y_1 and y_2 are solutions of the homogeneous second-order linear differential equation, and $y_1y_2' - y_2y_1' \neq 0$, then $y = c_1y_1 + c_2y_2$ is a general solution of the differential equation. This is the gist of the supplement. The expression $y_1y_2' - y_2y_1'$ is called a Wronksian.

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SOLUTIONS TO EVERY OTHER ODD EXERCISE

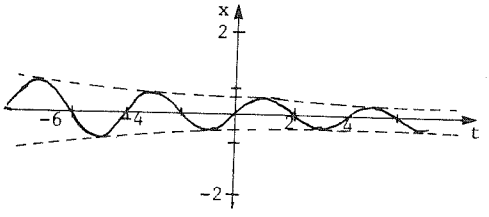
1. The general solution of $ay'' + by' + cy = 0$ is $y = c_1\exp(r_1x) + c_2\exp(r_2x)$, where r_1 and r_2 are the roots of $ar^2 + br + c = 0$ and c_1 and c_2 are constants. Here, $r = (4 \pm \sqrt{16 - 12})/2 = (4 \pm 2)/2 = 3$ or 1. Thus, the solution is $y = c_1\exp(3x) + c_2\exp(x)$.

5. Differentiating the solution to Exercise 1 gives $y' = 3c_1\exp(3x) + c_2\exp(x)$. Thus, $y(0) = 0$ and $y'(0) = 1$ gives $c_1 + c_2 = 0$ and $3c_1 + c_2 = 1$. This yields $c_1 = 1/2$ and $c_2 = -1/2$. The particular solution is $y = (1/2)\exp(3x) - (1/2)\exp(x)$.

9. We have $r = (4 \pm \sqrt{16 - 20})/2 = (4 \pm 2i)/2 = 2 \pm i$. Thus, the solution is $y = c_1\exp[(2 + i)x] + c_2\exp[(2 - i)x] = e^{2x}[c_1(\cos x + i \sin x) + c_2(\cos x - i \sin x)] = e^{2x}[c_1(\cos x + i \sin x) + c_2(\cos x - i \sin x)] = e^{2x}(a_1\cos x + a_2\sin x)$, where $a_1 = c_1 + c_2$ and $a_2 = i(c_1 - c_2)$.

13. The general solution is $y = (c_1 + c_2x)\exp(r_1x)$, where r_1 is the repeated root of the characteristic equation. Here, we have $r^2 - 6r + 9 = (r - 3)^2 = 0$, so $r = 3$ and the general solution is $y = (c_1 + c_2x)\exp(3x)$. Differentiation gives $y' = c_2\exp(3x) + (c_1 + c_2x) \times 3\exp(3x)$. Substituting $y(0) = 0$ and $y'(0) = 1$ yields $0 = c_1$ and $1 = c_2$. Thus, the particular solution is $y = \exp(3x)$.

17. (a) $\beta^2 - 4\omega^2 = \pi^2/256 - 4(\pi^2/4) < 0$, so the spring is underdamped.

(b) 

The general solution is $x = (c_1\cos \bar{\omega}t + c_2\sin \bar{\omega}t)\exp(-\pi t/32)$ where $\bar{\omega} = \sqrt{4\omega^2 - \beta^2}/2 = \sqrt{255}\pi/32$. $x' = (-c_1\bar{\omega}\sin \bar{\omega}t + c_2\bar{\omega}\cos \bar{\omega}t)\exp(-\pi t/32) + (c_1\cos \bar{\omega}t + c_2\sin \bar{\omega}t)(-\pi/32)\exp(-\pi t/32)$.

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17. (b) (continued)

Substituting $x(0) = 0$ and $x'(0) = 1$ yields $0 = c_1$ and $1 = c_2\bar{\omega}$, so $c_2 = 1/\bar{\omega}$. Thus, $x = (1/\bar{\omega})(\sin \bar{\omega}t)\exp(-\pi t/32)$.

21. The general solution is $y = y_p + y_h$, where y_h is the general solution of the homogeneous equation and y_p is a particular solution of the nonhomogeneous equation. The characteristic equation is $r^2 - 4r + 3 = (r - 3)(r - 1) = 0$, so the general solution of the homogeneous equation is $y = c_1\exp(3x) + c_2\exp(x)$. A particular solution should have the form $y = Ax + B$, so $y' = A$ and $y'' = 0$. Substitution gives $-4A + 3(Ax + B) = 3Ax + (-4A + 3B) = 6x + 10$; therefore, $A = 2$ and $B = 6$. Hence, the solution is $y = c_1\exp(3x) + c_2\exp(x) + 2x + 6$.

25. The characteristic equation is $r^2 - 4r + 5$. The roots are given by $(4 \pm \sqrt{16 - 20})/2 = 2 \pm 2i$, so the general solution of the characteristic equation is $y = c_1\exp[(2 + 2i)x] + c_2\exp[(2 - 2i)x] = e^{2x} \times [c_1(\cos 2x + i \sin 2x) + c_2(\cos 2x - i \sin 2x)] = e^{2x}(C_1\cos 2x + C_2\sin 2x)$, where $C_1 = c_1 + c_2$ and $C_2 = i(c_1 - c_2)$. A particular solution should have the form $y = Ax^2 + Bx + D$, so $y' = 2Ax + B$ and $y'' = 2A$. Substitution yields $2A - 4(2Ax + B) + 5(Ax^2 + Bx + D) = 5Ax^2 + (-8A + 5B)x + (2A - 4B + 5D) = x^2 + x$, i.e., $A = 1/5$, $B = 13/25$, and $D = 42/125$. Hence, the solution is $y = e^{2x}(C_1\cos 2x + C_2\sin 2x) + x^2/5 + 13x/25 + 42/125$.

29. The method of variation of parameters gives a particular solution, $y_p = v_1y_1 + v_2y_2$, where y_1 and y_2 are solutions of the homogeneous equation, and v_1 and v_2 are found by solving $v_1'y_1 + v_2'y_2 = 0$ and $v_1'y_1' + v_2'y_2' = F/a$. From Exercise 21, $y_1 = e^{3x}$ and $y_2 = e^x$, so we get $v_1'e^{3x} + v_2'e^x = 0$ and $v_1'(3e^{3x}) + v_2'e^x = (6x + 10)/1$. Subtraction

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29. (continued)

gives $2v_1'e^{3x} = 6x + 10$, i.e., $v_1' = (3x + 5)e^{-3x}$. Thus, by letting $u = 3x + 5$ and integrating by parts, we get $v_1 = -(3x + 5)e^{-3x}/3 + \int e^{-3x}dx = -(3x + 5)e^{-3x}/3 - e^{-3x}/3$. Similarly, $2v_2'e^{x} = -(6x + 10)$, i.e., $v_2' = -(3x + 5)e^{-x}$. Integration by parts with $u = -(3x + 5)$ yields $(3x + 5)e^{-x} - \int 3e^{-x}dx = (3x + 5)e^{-x} + 3e^{-x}$. Therefore, the general solution is $y = c_1\exp(3x) + c_2\exp(x) + [(-x - 2)\exp(-3x)] \times \exp(3x) + [(3x + 8)\exp(-x)]\exp(x) = c_1\exp(3x) + c_2\exp(x) + 2x + 6$.

33. From Exercise 21, $y_1 = \exp(3x)$ and $y_2 = \exp(x)$, so $v_1'\exp(3x) + v_2'\exp(x) = 0$ and $3v_1'\exp(3x) + v_2'\exp(x) = \tan x$. Subtraction yields $2v_1'\exp(3x) = \tan x$, i.e., $v_1' = (\tan x)\exp(-3x)/2$, and so $v_1 = \int(\tan x)\exp(-3x)dx/2$. Similarly, we get $2v_2'\exp(x) = -\tan x$, i.e., $v_2' = (-\tan x)\exp(-x)/2$, and so $v_2 = \int(\tan x)\exp(-x)dx/2$. Thus, the solution is $y = c_1\exp(3x) + c_2\exp(x) + [\exp(3x)/2]\int(\tan x)\exp(-3x)dx - [\exp(x)/2]\int(\tan x)\exp(-x)dx$.

37. Use the method of Example 7. Try a particular solution of the form $x = C\cos t$, so $x' = -C\sin t$, and $x'' = -C\cos t = -x$. Substitution yields $3C\cos t = 3\cos t$, so $C = 1$. The solution of the homogeneous equation is $x = A\cos 2t + B\sin 2t$, so a general solution is $x = A\cos 2t + B\sin 2t + \cos t$. Differentiation yields $x' = -2A\sin 2t + 2B\cos 2t - \sin t$; therefore, $x(0) = 0$ gives $0 = A + 1$ and $x'(0) = 0$ gives $0 = 2B$, i.e., $A = -1$ and $B = 0$. Hence, the solution is $x = -\cos 2t + \cos t = 2\sin(3t/2)\sin(t/2)$ by the product formula.

41. (a) Here, $r^2 + 4r + 25 = 0$ implies $r = (-4 \pm \sqrt{16-100})/2 = -2 \pm i\sqrt{21}$, and $F_0 = 2$, $m = 1$, $k = 25$, $\omega = 5$, $\Omega = 2$, $\gamma = 4$, and $\delta = \tan^{-1}(8/21)$. In terms of sine and cosine, the solution is $x(t) = \exp(-4t)[A\sin\sqrt{21}t + B\cos\sqrt{21}t] + (2/\sqrt{505})\cos(2t - \delta)$.

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41. (a) (continued)

Differentiation gives $x'(t) = -4\exp(-4t)[A \sin \sqrt{21}t + B \cos \sqrt{21}t] + \exp(-4t)[\sqrt{21} A \cos \sqrt{21}t - \sqrt{21} B \sin \sqrt{21}t] - 2(2/\sqrt{505})\sin(2t - \delta)$. $x(0) = 0$ gives $0 = B + (2/\sqrt{505})\cos \delta = B + (2/\sqrt{505})(21/\sqrt{505}) = B + 42/505$, i.e., $B = -42/505$; $x'(0) = 0$ gives $0 = -4B + \sqrt{21} A + (4/\sqrt{505})(8/\sqrt{505}) = 168/505 + 32/505 + \sqrt{21} A$, i.e., $A = -200/505\sqrt{21} = -40/101\sqrt{21}$. Thus, $x(t) = \exp(-4t)[(-40/101\sqrt{21}) \times \sin\sqrt{21} t - (42/505)\cos\sqrt{21} t] + (2/\sqrt{505})\cos(2t - \tan^{-1}(8/21))$.

(b) For large t , the graph approximates $(2/\sqrt{505})\cos(2t - \tan^{-1}(8/21))$.

45. If the Wronksian does not vanish, then y_1 and y_2 form a fundamental set. $y_1'(x) = r_1\exp(r_1x)$ and $y_2'(x) = \exp(r_1x) + xr_1\exp(r_1x)$. Thus, the Wronksian is $y_1(x)y_2'(x) - y_2(x)y_1'(x) = (1 + r_1x)\exp(2r_1x) - r_1x \exp(2r_1x) = \exp(2r_1x) \neq 0$. Therefore, y_1 and y_2 form a fundamental set and the supplement tells us that $c_1y_1 + c_2y_2$ is the general solution.

49. (a) If $y(x)$ is a solution, then, by the method of reduction of order, $v(x)y(x)$ is also a solution. By expression (18), with $y_1(x) = y(x)$ and $y_2(x) = v(x)y(x)$, we get $W(x) = y(x) \times [v'(x)y(x) + v(x)y'(x)] - v(x)y(x)y'(x) = [y(x)]^2v'(x)$. We assumed that $y(x) \neq 0$, so $W(x) = 0$ if and only if $v'(x) = 0$. However, if $v'(x) = 0$, then $v(x)$ is a constant and $v(x)y(x)$ is simply a mulitple of $y(x)$. Thus, $v'(x) \neq 0$ if $v(x)y(x)$ is another solution. Therefore, $[y(x)]^2v'(x) \neq 0$ and the fundamental set is $y(x)$ and $v(x)y(x)$.

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49. (b) We substitute $y_1(x) = x^r$ and $y_2(x) = (\ln x)x^{(1-\alpha)/2}$ into expression (18) to get $W(x) = x^r\{(1/x)x^{(1-\alpha)/2} + (\ln x) \times [(1 - \alpha)/2]x^{(-1-\alpha)/2}\} - (\ln x)x^{(1-\alpha)/2}(rx^{r-1}) = [(\ln x)(1 - \alpha)/2 + (1 - r \ln x)]x^{r+(-1-\alpha)/2}$. Now, by assumption, $r^2 + (\alpha - 1)r + \beta = 0$, so $r = ((1 - \alpha) \pm \sqrt{(\alpha - 1)^2 - 4\beta})/2$, and so $W(x) = [1 \pm (\ln x)\sqrt{(\alpha - 1)^2 - 4\beta}/2]x^{-\alpha \pm \sqrt{(\alpha-1)^2-4\beta}/2} \neq 0$. Therefore, x^r and $(\ln x)x^{(1-\alpha)/2}$ form a fundamental set.

The general solution for Euler's equation is $y(x) = c_1x^r + c_2(\ln x)x^{(1-\alpha)/2}$. The general solution for part (a) is $c_1y(x) + c_2v(x)y(x)$.

53. We expect a solution of the form $y = e^{rx}$, so $y' = re^{rx}$, $y'' = r^2e^{rx}$, $y''' = r^3e^{rx}$, and $y'''' = r^4e^{rx}$. Therefore, $y'''' + y = r^4e^{rx} + e^{rx} = 0 = (r^4 + 1)e^{rx}$. Solving $r^4 + 1 = 0$ yields $r = (1 + i)/\sqrt{2}$, $(1 - i)/\sqrt{2}$, $(-1 + i)/\sqrt{2}$, and $(-1 - i)/\sqrt{2}$. Therefore, the general solution is $e^{x/\sqrt{2}}(C_1\cos(x/\sqrt{2}) + C_2\sin(x/\sqrt{2})) + e^{x/\sqrt{2}}(C_3\cos(-x/\sqrt{2}) + C_4\sin(-x/\sqrt{2})) + e^{-x/\sqrt{2}}(C_5\cos(x/\sqrt{2}) + C_6\sin(x/\sqrt{2})) + e^{-x/\sqrt{2}}(C_7\cos(-x/\sqrt{2}) + C_8\sin(-x/\sqrt{2}))$. Since $\cos(x/\sqrt{2}) = \cos(-x/\sqrt{2})$ and $\sin(x/\sqrt{2}) = -\sin(x/\sqrt{2})$, the solution reduces to $y = e^{x/\sqrt{2}}(c_1\cos(x/\sqrt{2}) + c_2\sin(x/\sqrt{2})) + e^{-x/\sqrt{2}}(c_3\cos(x/\sqrt{2}) + c_4\sin(x/\sqrt{2}))$.

SECTION QUIZ

1. Find a general solution to each of the following differential equations:

(a) $2y'' - 2y' + y = 0$

(b) $y'' + 6y' + 9y = 0$

(c) $-2y'' + 3y' + 2y = 0$

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2. For each of the following differential equations, guess the general form of the particular equation:

   (a) $y'' + y' + y = \cos 2x + 3$

   (b) $y'' + y' + y = x^5 - x^3 + \tan x$

   (c) $y'' + y' + y = e^{-2x} + x^2 - 2$

3. Find the general solution of $y''' - y = x$. [Hint: one solution of the homogeneous equation should be $y = \exp(rx)$.]

4. Solve $y'' - 3y' + 2y = (1 - x^2)^{-3}$ and leave your answer as an integral.

5. Francis, the fruit fly, expends energy when he accelerates; however, he gains energy by speeding between fruit trees and covering more distance since he gets more fruit juices more rapidly. Thus, his distance can be described by $-y'' + 2y' + y = 0$, where $y(0) = 0$ and $y'(0) = \sqrt{2}$. If the distance function y is a function of time t, and the average tree is 0.5 units apart, how many trees has Francis the fruit fly visited after one unit of time (assuming he begins at $t = 0$)?

ANSWERS TO PREREQUISITE QUIZ

1. $x = (-b \pm \sqrt{b^2 - 4ac})/2a$

2. $5\sqrt{2}\exp(i\pi/4)$

3. $\cos x + i \sin x$

4. $y = 3\cos 2t + \sin 2t$

5. (a) $e^{3t}/3 + C$

   (b) $-\sin y + \cos y$

   (c) $x^3/3 + x^2 + C$

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ANSWERS TO SECTION QUIZ

1. (a) $[C_1 \sin(x/2) + C_2 \cos(x/2)] \exp(x/2)$

   (b) $(c_1 + c_2 x)\exp(3x)$

   (c) $c_1 \exp(2x) + c_2 \exp(-x/2)$

2. (a) $A \cos 2x + B \sin 2x + C$

   (b) $Ax^5 + Bx^4 + Cx^3 + Dx^2 + E + F \cos x + G \sin x$

   (c) $A \exp(-2x) + Bx^2 + Cx + D$

3. $c_1 e^x + [c_2 \cos(\sqrt{3}x/2) + c_3 \sin(\sqrt{3}x/2)] e^{-x/2} - x$.

4. $c_1 e^{2x} + c_2 e^x + e^{2x} \int [dx/e^{2x}(1 - x^2)^3] - e^x \int [dx/e^x(1 - x^2)^3]$

5. 42 trees

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12.8 Series Solutions of Differential Equations

PREREQUISITES

1. Recall how to solve a second-order linear differential equation of the form $ay'' + by' + cy = 0$ (Section 12.7).
2. Recall how to differentiate a power series (Section 12.4).

PREREQUISITE QUIZ

1. Find the general solution $y(x)$ of $y'' + 3y' + 2y = 0$.
2. Solve $y'' - 2y' + y = 0$, $y'(0) = 0$, $y(0) = 1$.
3. Find $(d/dx)\Sigma_{i=0}^{\infty}[(2x)^i/i!]$.
4. Find $(d/dy)\Sigma_{i=0}^{\infty}[(-y)^i/(i + 2)]$.

GOALS

1. Be able to solve differential equations by using power series.

STUDY HINTS

1. Series solution. Here we seek the solution of a differential equation in the form $y = \Sigma_{i=0}^{\infty}a_i x^i$. The a_i's need to be determined. Differentiate the series and substitute into the original equation. Be sure you have changed the summation index when you differentiated. At this point, write out the first few terms and look for a pattern. Finally, the ratio test may be used to show convergence.
2. Special equations. Legendre's and Hermite's equation are special in that they are specific equations that arise from physical problems. However, the method of solution is the same.

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3. Frobenius method. If the coefficient of y'' vanishes at $x = 0$, then look for a solution of the form $y = x^r\Sigma_{i=0}^{\infty}a_ix^i$. The method of solution is the same except that we now solve for the coefficient of x^{r-1}, x^r, x^{r+1}, etc., rather than x, x^2, x^3, etc. We solve for r, which is generally not an integer. See Example 5.

4. Indicial equation. The values of r in the Frobenius method is determined by setting the coefficient of the lowest power of x equal to zero. This is called the indicial (pronounced "in dish al") equation.

5. Repeated indicial roots. If the roots repeat, then the solutions have the form $y_1(x) = x^r\Sigma_{i=0}^{\infty}a_ix^i$ and $y_2(x) = y_1(x)\ln x + x^r\Sigma_{i=0}^{\infty}b_ix^i$.

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. Let $y = \Sigma_{i=0}^{\infty}a_ix^i$, so $y' = \Sigma_{i=1}^{\infty}ia_ix^{i-1}$ and $y'' = \Sigma_{i=2}^{\infty}i(i-1)a_ix^{i-2}$. Then equate the coefficients of x^i to 0. $y'' - xy' - y = \Sigma_{i=2}^{\infty}i(i-1)a_ix^{i-2} - \Sigma_{i=1}^{\infty}ia_ix^i - \Sigma_{i=0}^{\infty}a_ix^i = \Sigma_{i=0}^{\infty}(i+2)(i+1)a_{i+2}x^i - \Sigma_{i=1}^{\infty}ia_ix^i - \Sigma_{i=0}^{\infty}a_ix^i = 0$. Thus, $2a_2 - a_0 = 0$ (constant term), $6a_3 - a_1 - a_1 = 0$ (x term), $12a_4 - 2a_2 - a_2 = 0$ (x^2 term), and in general, $(i+2)(i+1)a_{i+2} - ia_i - a_i = 0$ (x^i term). Hence, $a_2 = a_0/2$, $a_3 = a_1/3$, $a_4 = a_2/4 = a_0/8$, $a_5 = a_3/5 = a_1/15$, and in general, $a_{i+2} = (i+1)a_i/(i+2)(i+1) = a_i/(i+2)$, i.e., $a_{2n} = a_0/2\cdot 4\cdot\ldots\cdot 2n = a_0/2^n(n!)$ and $a_{2n+1} = a_1/3\cdot 5\cdot\ldots\cdot(2n+1) = a_1 2^n(n!)/(2n+1)!$ Therefore, the solution is $y = a_0[\Sigma_{n=0}^{\infty}(x^n/2^n(n!))] + a_1[\Sigma_{n=0}^{\infty}(2^n(n!)x^{2n+1}/(2n+1)!]$.

5. Let $y = \Sigma_{i=0}^{\infty}a_ix^i$, so $y' = \Sigma_{i=1}^{\infty}ia_ix^{i-1}$ and $y'' = \Sigma_{i=2}^{\infty}i(i-1)a_ix^{i-2}$. $y(0) = 0$ and $y'(0) = 1$ implies $a_0 = 0$ and $a_1 = 1$. Now, $y'' + 2xy' = \Sigma_{i=2}^{\infty}i(i-1)a_ix^{i-2} + \Sigma_{i=1}^{\infty}2ia_ix^i = \Sigma_{i=0}^{\infty}(i+2)(i+1)a_{i+2}x^i + \Sigma_{i=1}^{\infty}2ia_ix^i = 0$. Thus, $2a_2 = 0$ (constant term), $6a_3 + 2a_1 = 0$ (x term), $12a_4 + 4a_2 = 0$ (x^2 term), and $20a_5 + 6a_3 = 0$ (x^3 term).

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5. (continued)

Hence, $a_2 = 0$, $a_3 = -a_1/3 = -1/3$, $a_4 = -a_2/3 = 0$, $a_5 = -3a_3/10 = 1/10$. Therefore, the solution is $y = x - x^3/3 + x^5/10 - \ldots$.

9. Let $y = \sum_{i=0}^{\infty} a_i x^i$, so $y' = \sum_{i=1}^{\infty} i a_i x^{i-1}$ and $y'' = \sum_{i=2}^{\infty} i(i-1)a_i x^{i-2}$. $y'' - xy = 0 = \sum_{i=2}^{\infty} i(i-1)a_i x^{i-2} - \sum_{i=0}^{\infty} a_i x^{i+1} = 2a_2 + \sum_{i=1}^{\infty}[(i+2)\times (i+1)a_{i+2}x^i - a_{i-1}x^i]$. Thus, $2 \cdot 1 a_2 = 0$ (constant term) , $3 \cdot 2 a_3 - a_0 = 0$ (x term) , $4 \cdot 3 a_4 - a_1 = 0$ (x^2 term) , $5 \cdot 4 a_5 - a_2 = 0$ (x^3 term) , $6 \cdot 5 a_6 - a_3 = 0$ (x^4 term) , $7 \cdot 6 a_7 - a_4 = 0$ (x^5 term) , and in general $(i+2)(i+1)a_{i+2} - a_{i-1}$ (x^i term). Hence, $a_2 = 0$, $a_3 = a_0/6$, $a_4 = a_1/12$, $a_5 = a_2/20 = 0$, $a_6 = a_3/30 = a_0/180$, $a_7 = a_4/42 = a_1/504$, and in general, $a_{i+3} = a_i/(i+3)(i+2)$. Therefore, the solution is $y = a_0(1 + x^3/6 + x^6/180 + \ldots) + a_1(x + x^4/12 + x^7/504 + \ldots)$. The recursion formula is $a_{i+3} = a_i/(i+3)(i+2)$

13. Let $y = a_0x^r + a_1x^{r+1} + a_2x^{r+2} + \ldots$, so $y' = ra_0x^{r-1} + (r+1)a_1x^r + (r+2)a_2x^{r+1} + \ldots$ and $y'' = r(r-1)a_0x^{r-2} + r(r+1)a_1x^{r-1} + (r+2)\times (r+1)a_2x^r + \ldots$. Thus, $3x^2y'' + 2xy' + y = [3r(r-1)a_0 + 2ra_0 + a_0]x^r + [3r(r+1)a_1 + 2(r+1)a_1 + a_1]x^{r+1} + \ldots = 0$. Setting the coefficient of x^r equal to 0 yields $a_0(3r^2 - r + 1) = 0$, i.e., $r = (1 \pm \sqrt{11}i)/6$. For the x^{r+1} coefficient, we set $a_1(3r^2 + 5r + 3) = 0$, which yields $a_1 = 0$ because r must be $(1 \pm \sqrt{11}i)/6$. Similarly, $a_2 = a_3 = \ldots = 0$. Thus, $y = a_0x^{(1+\sqrt{11}i)/6}$ is one solution and $y = a_0x^{(1-\sqrt{11}i)/6}$ is another solution. The general solution is $y = c_1x^{(1+\sqrt{11}i)/6} + c_2x^{(1-\sqrt{11}i)/6}$. Note that $x^{(1+\sqrt{11}i)/6} = e^{(\ln x)(1+\sqrt{11}i)/6} = e^{(\ln x)/6}e^{(\sqrt{11}\ln x)i/6} = x^{1/6}(\cos(\sqrt{11}\ln x/6) + i\sin(\sqrt{11}\ln x/6))$. Therefore, an equivalent solution is $x^{1/6}[b_1\cos(\sqrt{11}\ln x/6) + b_2\sin(\sqrt{11}\ln x/6)]$.

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17. Let $y = \Sigma_{n=0}^{\infty} a_n x^n$, so $y' = \Sigma_{n=1}^{\infty} n a_n x^{n-1}$, and $y'' = \Sigma_{n=2}^{\infty} [n(n-1) \times a_n x^{n-2}]$. Thus, $y'' + \omega^2 y = \Sigma_{j=0}^{\infty} [(j+2)(j+1)a_{j+2}x^j + \omega^2 a_j x^j]$. If $j = 0$, then $2a_2 + \omega^2 a_0 = 0$ or $a_2 = -\omega^2 a_0/2$. If $j = 1$, then $6a_3 + \omega^2 a_1 = 0$ or $a_3 = -\omega^2 a_1/6$. If $j = 2$, then $12a_4 + \omega^2 a_2 = 0$ or $a_4 = -\omega^2 a_2/12 = \omega^4 a_0/24$. If $j = 3$, then $20a_5 + \omega^2 a_3 = 0$ or $a_5 = -\omega^2 a_3/20 = \omega^4 a_1/120$. In general, $a_{2n} = (-1)^{2n}\omega^{2n} a_0/(2n)!$ and $a_{2n+1} = (-1)^{2n}\omega^{2n} a_1/(2n+1)!$. Thus, the general solution is $y = a_0(1 - \omega^2 x^2/2! + \omega^4 x^4/4! - \ldots) + a_1(x - \omega^2 x^3/3! + \omega^4 x^5/5! - \ldots)$. We recognize $1 - \omega^2 x^2/2! + \omega^4 x^4/4! - \ldots$ as the Maclaurin series of $\cos \omega x$, and we recognize $x - \omega^2 x^3/3! + \omega^4 x^5/5! - \ldots$ as the Maclaurin series of $\sin \omega x/\omega$. Thus, the solution is $y = A \cos \omega x + B \sin \omega x$, where $A = a_0$ and $B = a_1/\omega$.

21. In Example 3, we let $y_1 = 1 - (\lambda/2)x^2 - [(6 - \lambda)/4 \cdot 3 \cdot 2]x^4 + \ldots$ and we let $y_2 = x + [(2 - \lambda)/3 \cdot 2]x^3 + [(12 - \lambda)(2 - \lambda)/5 \cdot 4 \cdot 3 \cdot 2]x^5 + \ldots$. Thus, $y_1' = -\lambda x - [(6 - \lambda)/3 \cdot 2]x^3 + \ldots$ and $y_2' = 1 + [(2 - \lambda)/2]x^2 + [(12 - \lambda)(2 - \lambda)/4 \cdot 3 \cdot 2]x^4 + \ldots$. Since all of the series do converge, they can be multiplied, so the Wronskian is $W(x) = y_1(x)y_2'(x) - y_2(x)y_1'(x) = 1 + x^2 + (18 - 25\lambda + 7\lambda^2)x^4/24 + \ldots - [-\lambda x^2 + (\lambda^2 - \lambda - 6) \times x^4/6 + \ldots] = 1 + (1 + \lambda)x^2 + \ldots$. Note that all of the higher terms have even exponents, so $W(x) \geqslant 1 \neq 0$. Therefore, y_1 and y_2 form a fundamental set.

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SECTION QUIZ

1. Find the first few terms of the power series solutions for $y''' - x^2y'' + y = 0$.

2. (a) Find a power series solution for $y'' - 2y' = 0$.

   (b) Use the methods of Section 12.7 to solve $y'' - 2y' = 0$.

   (c) What special equation do you get by equating your answers to parts (a) and (b)?

3. An obnoxious travelling salesman has made you extremely irritated. Consequently, he irks you into slamming the spring door into his face. Due to a defect, the force exerted by the spring is given by $2y'' - 3y' + xy = 0$, where y is a function of x , the door's position. Find a power series for $y(x)$ if $y(0) = y'(0) = 1$

ANSWERS TO PREREQUISITE QUIZ

1. $y = c_1\exp(-2x) + c_2\exp(-x)$.
2. $y = (1 - x)e^x$
3. $\Sigma_{i=1}^{\infty}[2^i(2x)^{i-1}/(i - 1)!]$
4. $\Sigma_{i=1}^{\infty}[i(-1)^i(-y)^{i-1}/(i + 2)]$

ANSWERS TO SECTION QUIZ

1. $a_0(1 - x^3/6 - 5x^6/720 - \ldots) + a_1(x - x^4/24 - 11x^7/5040 - \ldots) + a_2(x^2 + x^5/60 + 19x^8/20160 + \ldots)$
2. (a) $a_0\Sigma_{n=0}^{\infty}[(2x)^n/n!]$

   (b) $y = c_1\exp(2x) + c_2$

   (c) $e^{2x} = \Sigma_{n=0}^{\infty}[(2x)^n/n!]$
3. $1 + x - 3x^2/4 + 7x^3/24 - 87x^4/576 + \ldots$

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12.R Review Exercises for Chapter 12

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. This is a geometric series, so it converges. $\Sigma_{i=1}^{\infty}(1/12^i) = \Sigma_{i=0}^{\infty}(1/12)\times (1/12)^i = (1/12)/(1 - 1/12) = 1/11$.

5. $1 + 2 + 1/3 + 1/3^2 + 1/3^3 + \ldots = 2 + \Sigma_{i=0}^{\infty}(1/3)^i = 2 + 1/(1 - 1/3) = 2 + 3/2 = 7/2$, so it converges.

9. $\Sigma_{n=1}^{\infty}5^{-n} = \Sigma_{n=1}^{\infty}(1/5)^n$ converges since it is a geometric series with $r < 1$.

13. Let $a_n = (-1)^n n/3^n$. Then, $\lim_{n\to\infty}|a_n/a_{n-1}| = \lim_{n\to\infty}\{[n/(n - 1)]/3\} = 1/3$. Therefore, $\Sigma_{n=1}^{\infty}a_n$ converges by the ratio test.

17. Let $a_n = 2^{(n2)}/n!$. Then $\lim_{n\to\infty}|a_n/a_{n-1}| = \lim_{n\to\infty}[2^{n^2-(n-1)^2}/n] = \lim_{n\to\infty}(2^{2n-1}/n) = \infty$. Therefore, $\Sigma_{n=1}^{\infty}(2^{(n^2)}/n!)$ diverges by the ratio test.

21. Let $a_n = 1/(\ln n)^{\ln n}$, so $\ln a_n = -\ln n(\ln \ln n)$. Since $\ln n$ is an increasing function, $\ln \ln n \geqslant 2$ if $n \geqslant e^{(e^2)}$. Thus, for such n , $a_n \leqslant 1/n^2$, so the series converges by comparison with $b_n = 1/n^2$.

25. The error made in estimating the sum of an alternating series is no greater than $|a_{n+1}|$. $1 - 1/4 + 1/16 - 1/32 + \ldots \approx 0.78$. Although there is no formula for the general term, we note that $1/32 < 0.05$, so the sum is approximately 0.78 .

29. $r = |a_n/a_{n-1}| = |(1 - n)3^{n-1}/(2 - n)3^n| = |(1 - n)/(2 - n)3|$. We want to find N such that $|[(1 - N)/3^N]r/(1 - r)| < 0.05$. The error is less than $(1 - N^2)/3^N(5 - 2N)$, which is less than 0.05 for $N \geqslant 5$. Thus, $\Sigma_{n=1}^{\infty}[(1 - n)/3^n] \approx 0 - 1/9 - 2/27 - 3/81 - 4/243 - \ldots \approx -0.24$.

33. False; consider $\Sigma_{n=1}^{\infty}(1/n)$.

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37. False; $e^{2x} = \Sigma_{i=0}^{\infty}[(2x)^i/i!] = 1 + 2x + 2x^2 + 4x^3/3 + \ldots$.

41. True; by the sum rule, $\Sigma_{j=1}^{\infty}a_j + \Sigma_{k=0}^{\infty}b_k = b_0 + \Sigma_{j=1}^{\infty}a_j + \Sigma_{k=1}^{\infty}b_k = b_0 + \Sigma_{i=1}^{\infty}(a_i + b_i)$.

45. True; $|a_n| \leq |a_n| + |b_n|$. Therefore, by the comparison test, the convergence of $\Sigma_{n=1}^{\infty}(|a_n| + |b_n|)$ implies the convergence of $\Sigma_{n=1}^{\infty}|a_n|$.

49. This is a geometric series, so $\Sigma_{n=1}^{\infty}(1/9^n) = \Sigma_{n=0}^{\infty}(1/9)(1/9)^n = (1/9)/(1 - 1/9) = 1/8$.

53. Let $a_i = (-1)^i x^{2i}/(i + 1)!$. Then $\lim_{i\to\infty}|a_i/a_{i-1}| = \lim_{i\to\infty}[x^2/(i + 1)] = 0$. Therefore, the radius of convergence is ∞ by the ratio test.

57. Let $a_n = (-1)^n/2^n$. Then $\ell = \lim_{n\to\infty}|a_n/a_{n-1}| = 1/2$. Therefore, the radius of convergence is $1/\ell = 2$.

61. Since $\ln(1 + x) = \Sigma_{i=1}^{\infty}[(-1)^{i+1}x^i/i]$, we have $\ln(1 + x^4) = \Sigma_{i=1}^{\infty}[(-1)^{i+1}x^{4i}/i]$.

65. We have $e^t = \Sigma_{i=0}^{\infty}(t^i/i!)$, so $e^t - 1 = \Sigma_{i=1}^{\infty}(t^i/i!)$. Then $(e^t - 1)/t = \Sigma_{i=1}^{\infty}(t^{i-1}/i!)$; therefore, $\int_0^x[(e^t - 1)/t]dt = \Sigma_{i=1}^{\infty}[x^i/i(i!)]$.

69. If $f(x) = x^{3/2}$, then $f'(x) = (3/2)x^{1/2}$, $f''(x) = (3/2)(1/2)x^{-1/2}$, $f'''(x) = (3/2)(1/2)(-1/2)x^{-3/2}$, and in general, $f^{(n)}(x) = [(3)(1)(-1)\cdot\ldots\cdot(3 - 2(n - 1))/2^n]x^{(3-2n)/2}$. Evaluating at $x = 1$, we get the coefficients since one to any power is one, i.e., $f(1) = 1$; $f'(1) = 3/2$; $f''(1) = 3/4$; ... $f^{(n)}(1) = (3)(1)(-1) \ldots (5 - 2n)/2^n$. Therefore, the Taylor expansion of $x^{3/2}$ about $x = 1$ is $1 + (3/2)(x - 1) + [(3/4)/2!](x - 1)^2 + [f^{(n)}(1)/n!](x - 1)^n = \Sigma_{n=0}^{\infty}[(3)(1)(-1)\cdot\ldots\cdot(5 - 2n)/2^n n!](x - 1)^n$. Alternatively, one may use the binomial series $(1 + u)^{\alpha} = \Sigma_{i=0}^{\infty}[\alpha(\alpha - 1)\cdots(\alpha - i + 1)u^i/i!]$ with $\alpha = 3/2$ and $u = x - 1$.

By the ratio test, $|a_n/a_{n-1}| = |[(3 - 2n)(5 - 2n)/2^n n!][2^{n-1}(n - 1)!/$

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69. (continued)

$(3 - 2n)]| = |(5 - 2n)/2n|$. The radius of convergence is the reciprocal of $\lim_{n\to\infty}|a_n/a_{n-1}| = 1$, so $R = 1$.

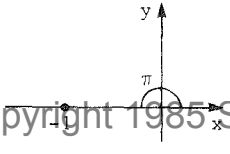
73. The Maclaurin series for $(1 + x)^{3/2}$ is $1 + 3x/2 + 3x^2/8 - x^3/16 + 3x^4/128 - 3x^5/256 + \ldots$ and $(1 - x)^{3/2}$ is $1 - 3x/2 + 3x^2/8 + x^3/16 + 3x^4/128 + 3x^5/256 + \ldots$. Therefore, $(1 + x)^{3/2} - (1 - x)^{3/2} = 3x - x^3/8 - 3x^5/128 - \ldots$. Dividing by x gives $3 - x^2/8 - 3x^4/128 - \ldots$, so the limit as x approaches 0 is 3 .

77. The real part of $a + bi$ is a ; the imaginary part is b ; the complex conjugate is $a - bi$; and the absolute value is $\sqrt{a^2 + b^2}$. If $\sqrt{2 - i} = a + bi$, then $2 - i = (a + bi)^2 = a^2 - b^2 + 2abi$. Thus, $a^2 - b^2 = 2$ and $2ab = -1$, i.e., $b = -1/2a$ and so $a^2 - 1/4a^2 = 2$, i.e., $4a^4 - 1 = 8a^2$. Rearrangement gives $4a^4 - 8a^2 = 1$ and completing the square gives $4(a^4 - 2a^2 + 1) = 5 = 4(a^2 - 1)^2$, i.e., $\pm\sqrt{5/4} = a^2 - 1$, so $a = \pm(\pm\sqrt{5/4} + 1)^{1/2}$. $b = -1/2a$, so $\sqrt{2 - i} = (\sqrt{5/4} + 1)^{1/2} - (1/2(\sqrt{5/4} + 1)^{1/2})i$ and $(-\sqrt{5/4} + 1)^{1/2} - (1/2(-\sqrt{5/4} + 1)^{1/2})i$.

Thus, if $z = (\sqrt{5/4} + 1)^{1/2} - (1/2(\sqrt{5/4} + 1)^{1/2})i$, then the real part is $(\sqrt{5/4} + 1)^{1/2}$; the imaginary part is $-(1/2(\sqrt{5/4} + 1)^{1/2})$; the complex conjugate is $(\sqrt{5/4} + 1)^{1/2} + (1/2(\sqrt{5/4} + 1)^{1/2})i$; and the absolute value is $\sqrt[4]{5}$.

If $z = (-\sqrt{5/4} + 1)^{1/2} - (1/2(-\sqrt{5/4} + 1)^{1/2})i$, then the real part is $(-\sqrt{5/4} + 1)^{1/2}$; the imaginary part is $-(1/2(-\sqrt{5/4} + 1)^{1/2})i$; the complex conjugate is $(-\sqrt{5/4} + 1)^{1/2} + (1/2(-\sqrt{5/4} + 1)^{1/2})i$; and the absolute value is $\sqrt[4]{5}$.

81. $\exp(\pi i/2) = i$ and $i^2 = -1$, so $r = 1$ and $\theta = \pi$. Thus, $z = \exp(\pi i)$.



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85. The characteristic equation is $r^2 + 4 = 0$, which has the roots $\pm 2i$. Thus, the general solution is $y = c_1\exp(2ix) + c_2\exp(-2ix) = c_1(\cos 2x + i \sin 2x) + c_2(\cos(-2x) + i \sin(-2x)) = c_1(\cos 2x + i \sin 2x) + c_2(\cos 2x - i \sin 2x) = (c_1 + c_2)\cos 2x + i(c_1 - c_2)\sin 2x = C_1\cos 2x + C_2\sin 2x$, where $C_1 = c_1 + c_2$ and $C_2 = i(c_1 - c_2)$.

89. The characteristic equation is $r^2 + 3r - 10 = 0$, whose solution is given by $r = (-3 \pm \sqrt{9 + 40})/2 = (-3 \pm 7)/2 = -5$ or 2. Thus, the solution to the homogeneous equation is $y = c_1\exp(-5x) + c_2\exp(2x)$. Let $Ae^x + B \cos x + C \sin x = y$, so $y' = Ae^x - B \sin x + C \cos x$, and $y'' = Ae^x - B \cos x - C \sin x$. Therefore, $y'' + 3y' - 10y = -6Ae^x + (3C - 11B)\cos x + (-3B - 11C)\sin x = e^x + \cos x$, i.e., $-6A = 1$, $3C - 11B = 1$ and $-3B - 11C = 0$. Hence, $A = -1/6$, $C = -3B/11$, so $-9B/11 - 11B = -130B/11 = 1$, i.e., $B = -11/130$ and $C = 33/130$. Thus, the general solution of the nonhomogeneous equation is $y = -e^x/6 - 11 \cos x/130 + 33 \sin x/130 + c_1\exp(-5x) + c_2\exp(2x)$.

93. The characteristic equation for $y'' + 4y = 0$ is $r^2 + 4 = 0$, so $r = \pm 2i$. Thus, the general solution to the homogeneous equation is $y = c_1\cos 2x + c_2\sin 2x$. To obtain a particular solution, we use the method of variation of parameters. Thus, we have $y_1' = \cos 2x$ and $y_2' = \sin 2x$, and we must solve $v_1'\cos 2x + v_2'\sin 2x = 0$ and $-2v_1'\sin 2x + 2v_2'\cos 2x = x/\sqrt{x^2 + 1}$ simultaneously. This yields $v_1 = \int[-x \sin 2x/2\sqrt{x^2 + 1}]\,dx$ and $v_2 = \int[x \cos 2x/2\sqrt{x^2 + 1}]\,dx$. Therefore, the general solution is $y = c_1\cos 2x + c_2\sin 2x + \cos 2x\int[-x \sin 2x/2\sqrt{x^2 + 1}]\,dx + \sin 2x\int[x \cos 2x/2\sqrt{x^2 + 1}]\,dx$. Since the constants c_1 and c_2 are incorporated into the integrals, the solution simplifies to $y = -\cos 2x\int[x \sin 2x/2\sqrt{x^2 + 1}]\,dx + \sin 2x\int[x \cos 2x/2\sqrt{x^2 + 1}]\,dx$.

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97. The equation has the form of the damped forced oscillation equation which is discussed in the box on p. 628. Here, we have $m = 1$, $k = 9$, $\gamma = 1$, $F_0 = 1$, and $\Omega = 2$. The solution of the characteristic equation $r^2 + r + 9 = 0$ is $r = (-1 \pm \sqrt{35}\, i)/2$. Thus, the solution of the homogeneous equation is $e^{-t/2}(c_1\cos\sqrt{35}t + c_2\sin\sqrt{35}t)$. This transient part approaches zero as t approaches ∞.

For the particular equation, $\omega = \sqrt{k/m} = 3$ and $\delta = \tan^{-1}[\gamma\Omega/m(\omega^2 - \Omega^2)] = \tan^{-1}(2/5) \approx 0.38$. So $F_0\cos(\Omega t - \delta)/\sqrt{m^2(\omega^2 - \Omega^2)^2 + \gamma^2\omega^2} \approx \cos(2t - 0.38)/\sqrt{34}$. This is the limiting behavior as $t \to \infty$.

101. If $y = \Sigma_{n=0}^{\infty} a_n x^n$, then $y' = \Sigma_{n=1}^{\infty} n a_n x^{n-1}$, and $y'' = \Sigma_{n=2}^{\infty} n(n-1) \times a_n x^{n-2}$. Thus, $y'' + 2xy = \Sigma_{n=2}^{\infty} n(n-1)a_n x^{n-2} + 2\Sigma_{n=0}^{\infty} a_n x^{n+1} = 2a_2 + \Sigma_{j=1}^{\infty}[(j+2)(j+1)a_{j+2} + 2a_{j-1}]x^j = 0$. This gives us $a_2 = 0$. If $j = 1$, then $6a_3 + 2a_0 = 0$ or $a_3 = -a_0/3$. If $j = 2$, then $12a_4 + 2a_1 = 0$ or $a_4 = -a_1/6$. If $j = 3$, then $20a_5 + 2a_2 = 0$ or $a_5 = -a_2/10 = 0$. If $j = 4$, then $30a_6 + 2a_3 = 0$ or $a_6 = -a_3/15 = a_0/45$. Therefore, the general solution is $a_0(1 - x^3/3 + x^6/45 - \ldots) + a_1(1 - x^4/6 + \ldots)$.

105. If $y = \Sigma_{n=0}^{\infty} a_n x^n$, then $y' = \Sigma_{n=1}^{\infty} n a_n x^{n-1}$, and $y'' = \Sigma_{n=2}^{\infty} n(n-1) \times a_n x^{n-2}$. Thus, $5x^2y'' + y' + y = 0 = 5\Sigma_{n=2}^{\infty} n(n-1)a_n x^n + \Sigma_{n=1}^{\infty} n a_n x^{n-1} + \Sigma_{n=0}^{\infty} a_n x^n = (a_0 + a_1) + (a_1 + 2a_2)x + \Sigma_{j=2}^{\infty}[5j(j-1)a_j + (j+1)a_{j+1} + a_j]x^j$. This gives us $a_1 = -a_0$, $a_2 = -a_1/2 = a_0/2$. If $j = 2$, then $11a_2 + 3a_3 = 0$ or $a_3 = -11a_2/3 = -11a_0/6$. If $j = 3$, then $31a_3 + 4a_4 = 0$ or $a_4 = -31a_3/4 = 343a_0/24$. Thus, one solution is $a_0(1 - x + x^2/2 - 11x^3/6 + 343x^4/24 + \ldots)$.

109. (a) The equation may be rewritten as $L(d^2I/dt^2) = -I/C - R(dI/dt) + dE/dt$. Thus, according to the box on p. 628, this is a damped spring equation if $m = L$, $k = 1/C$, and $\gamma = R$.

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109. (b) The characteristic equation of $5(d^2I/dt^2) + 100(dI/dt) + I/0.1 = 0$ is $5r^2 + 100r + 10 = 0 = r^2 + 20r + 2$, so $r = (-20 \pm \sqrt{392})/2 = -10 \pm \sqrt{98}$. Thus, the solution of the homogeneous equation is $I = c_1\exp(-10 - \sqrt{98})t + c_2\exp(-10 + \sqrt{98})t$.

Now, $dE/dt = -120\pi \sin(60\pi t)$, so by the method of undetermined coefficients, we guess that a particular solution is $I_p = A \sin \Omega t + B \cos \Omega t$, where $\Omega = 60\pi$. The original equation is equivalent to $d^2I/dt^2 + 20(dI/dt) + 2I = -24\pi\sin(60\pi t)$. Differentiating I_p and substituting into the differential equation yields $(2A - 20B\Omega - A\Omega^2)\sin \Omega t + (2B + 20A\Omega - B\Omega^2)\cos \Omega t = -(2\Omega/5)\sin \Omega t$. Solving for A and B yields $A = 2\Omega(\Omega^2 - 2)/5[(\Omega^2 - 2)^2 + 400\Omega^2]$ and $B = 8\Omega^2/[(\Omega^2 - 2)^2 + 400\Omega^2]$. Therefore, the solution is $I = c_1\exp(-10 - \sqrt{98})t + c_2\exp(-10 + \sqrt{98})t + A \sin 60\pi t + B \cos 60\pi t$. Using $I(0) = 0$ yields $c_1 + c_2 + B = 0$ and using $I'(0) = 0$ yields $c_1(-10 - \sqrt{98}) + c_2(-10 + \sqrt{98}) + 60\pi A = 0$. Solving for c_1 and c_2 yields $c_1 = [60\pi A + (6 - \sqrt{98})B]/(-2 + 2\sqrt{98})$ and $c_2 = [-60\pi A - (8 + \sqrt{98})B]/(-2 + 2\sqrt{98})$. Therefore, $I(t) = 0.02217 \exp(-19.90t) - 0.02245 \exp(-0.1005t) + 0.002099 \sin 60\pi t + 0.0002227 \cos 60\pi t$.

113. (a) For any fixed value of x and t, let $S_n = \Sigma_{i=1}^{n} A_i \sin(i\pi x/L) \times \cos(i\pi ct/L)$. If the sequence of partial sums $S_1, S_2, \ldots$ converges to a limit, then the series converges for this value of x and t, and $\lim_{n\to\infty} S_n = y(x,t)$.

(b) For $t = 0$, $x = L/2$, the deflection of the string is $\Sigma_{n=1}^{\infty} A_n \sin[(n\pi L/2)/L] = \Sigma_{n=1}^{\infty} A_n \sin(n\pi/2) = \Sigma_{k=0}^{\infty} (-1)^k A_{2k+1}$.

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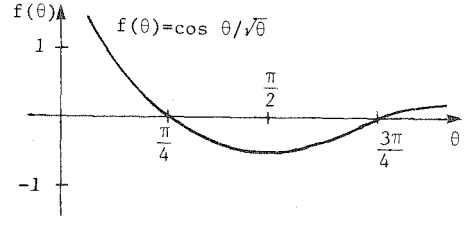
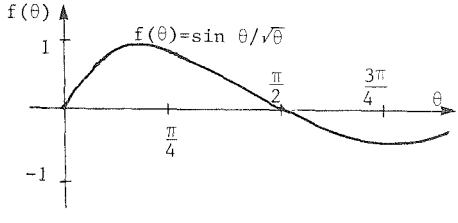
117. (a) Using the binomial series, with $\alpha = 1/2$, $x = 1/4$, $\sqrt{5/4} \approx 1 + (1/2)(1/4) + (1/2)(-1/2)(1/4)^2/2 \approx 1.12$.

(b) $\sqrt{5} = \sqrt{4(5/4)} = 2\sqrt{5/4} \approx 2(1.12) = 2.24$. This is accurate within $2(0.01) = 0.02$.

121. (a) Expanding, $\cos\theta/\sqrt{\theta} = \theta^{-1/2} - (1/2!)\theta^{3/2} + (1/4!)\theta^{7/2} - \ldots$ and $\sin\theta/\sqrt{\theta} = \theta^{1/2} - (1/3!)\theta^{5/2} + (1/5!)\theta^{9/2} - \ldots$. Thus, $\int_0^{\pi/4}(\cos\theta/\sqrt{\theta})d\theta = 2(\pi/4)^{1/2} - (2/5\cdot 2!)(\pi/4)^{5/2} + (2/9\cdot 4!) \times (\pi/4)^{9/2} - \ldots$ and $\int_0^{\pi/4}(\sin\theta/\sqrt{\theta})d\theta = (2/3)(\pi/4)^{3/2} - (2/7\cdot 3!) \times (\pi/4)^{7/2} + (2/11\cdot 5!)(\pi/4)^{11/2} - \ldots$. Taking the ratio and simplifying gives $x/y = (4/\pi)([1 - (1/5\cdot 2!)(\pi/4)^2 + (1/9\cdot 4!) \times (\pi/4)^4 - \ldots]/[1/3 - (1/7\cdot 3!)(\pi/4)^2 + (1/11\cdot 5!)(\pi/4)^4 - \ldots])$. Using just the leading terms, $x/y \approx (4/\pi)(1/(1/3)) = 12/\pi \approx 3.8$. Using the first two terms in the numerator and denominator gives $x/y \approx (4/\pi)(1 - (1/10)(\pi/4)^2)/(1/3 - (1/42)(\pi/4)^2) \approx 3.68$.

(b) The graphs of $\cos\theta/\sqrt{\theta}$ and $\sin\theta/\sqrt{\theta}$ are shown below. x and y are given parametrically as the area under these curves when $k = 1$.





The transitional spiral is drawn below for $k = 1$ and $0 \leq \phi \leq \pi$. As seen from the graphs above, x will continue to decrease until $\phi = 3\pi/2$. The y-value is just starting to decrease at $\phi = \pi$. Some of the points we have plotted are $(0,0)$ for $\phi = 0$, $(1.23,0.16)$ for $\phi = \pi/8$, $(1.66,0.44)$ for $\phi = \pi/4$,

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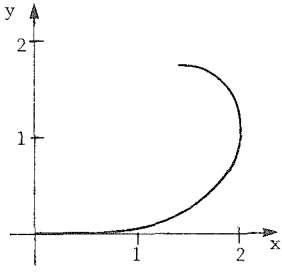
121. (b) (continued)



(1.95,1.09) for $\phi = \pi/2$, (1.75, 1.61) for $\phi = 3\pi/4$, and (1.31, 1.77) for $\phi = \pi$. We approximated the sine and cosine functions with the first five nonzero terms of the Maclaurin series.

125. True; $\Sigma_{n=1}^{\infty}[(a_n^2 + b_n^2)/2]$ converges by the sum rule. Since $a_n^2 - 2|a_n||b_n| + b_n^2 = (|a_n| - |b_n|)^2 \geq 0$, $|a_n||b_n| \leq (a_n^2 + b_n^2)/2$. Therefore, $\Sigma_{n=1}^{\infty}|a_n b_n|$ converges or $\Sigma_{n=1}^{\infty}a_n b_n$ converges absolutely.

129. Assume that $e = a/b$ for some integers a and b, so e is rational. Now, let $k > b$ and let $\alpha = k!(e - 2 - 1/2! - 1/3! - \ldots - 1/k!) = k!(a/b) - 2k! - k!/2! - k!/3! - \ldots - 1$. Since $k > b$, α is an integer. The quantity $e - 2 - 1/2! - 1/3! - \ldots - 1/k!$ is simply $e - \Sigma_{n=0}^{k}(1/n!)$. Using the Maclaurin expansion of e, α becomes $k![1/(k+1)! + 1/(k+2)! + \ldots] = (k!/k!)[1/(k+1) + 1/((k+2) \times (k+1)) + \ldots] < \Sigma_{n=1}^{\infty}(k+1)^{-n}$. The latter is a geometric series whose sum is $[1/(k+1)]/[1 - 1/(k+1)] = 1/k$. Thus, $\alpha < 1/k$, so α is not an integer, a contradiction. Therefore, e is irrational.

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TEST FOR CHAPTER 12

1. True or false.

   (a) If $\Sigma_{i=0}^{\infty} a_i$ diverges and $\Sigma_{i=0}^{\infty} b_i$ also diverges, then $\Sigma_{i=0}^{\infty}(a_i + b_i)$ also diverges.

   (b) The derivative of $\Sigma_{i=1}^{\infty}(x^{i+1}/i!)$ is $\Sigma_{i=1}^{\infty}[(i + 1)x^i/i!]$.

   (c) Any series that converges absolutely at $x = x_0$ also converges conditionally at that point.

   (d) Any equation of the form $at^2 + b(t + 2) - c = 0$, where a , b , and c are real constants, has at least one solution in the complex number system.

   (e) The series $\Sigma_{j=1}^{\infty}[(-1)^3/j]$ converges as an alternating series.

2. Suppose that $h(t)$ is a third degree polynomial and that $h(0) = -2$, $h'(0) = 8$, $h''(0) = -18$, $h'''(0) = 24$.

   (a) Find the Maclaurin series for $h(x)$.

   (b) Find the Taylor series expansion for h around $x = 1$.

   (c) Find an equivalent expression with the form $a(x - x_0)^3 + b(x - x_0) + c$, where a , b , c and x_0 are constants.

3. Find values of x for which $\Sigma_{n=5}^{\infty} n^4(x + 1)^n$:

   (a) converges absolutely

   (b) converges conditionally

   (c) diverges

4. Discuss the convergence or divergence of $\Sigma_{n=1}^{\infty}(a_n n)^{-1}$ if:

   (a) a_n is the last digit of n , i.e., $a_{13} = 3$, $a_{145} = 5$, etc.

   (b) $a_n = n$

5. (a) Find a solution for $w(x)$ which satisfies $w'' + 4w = \cos 2x + e^x$. (Hint: A particular solution has the form $w = Ax \cos 2x + Bx \sin 2x + Ce^x$.)

   (b) Solve $y''' + 2y'' + y' = 0$ for $y(x)$.

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6. Do the following situations imply convergence, divergence, or give no information? Unless otherwise specified, the series we are testing is $\Sigma_{i=1}^{\infty} a_i$, where a_i is assumed to be finite for all $i \geqslant 1$.

(a) $\lim_{i \to \infty} a_i = 0$

(b) $a_i = i^{-1.1}$

(c) $\lim_{i \to \infty} |a_i / a_{i-1}| = 1$ and the signs are alternating.

(d) $\lim_{i \to \infty} |a_i / a_{i+1}| > 1$ and every third sign is negative.

(e) $\lim_{i \to \infty} (1/a_i)^{1/i} = 1/2$

7. (a) Find the fourth degree Taylor polynomial for $f(x) = \tan x$ expanded around $x = \pi$.

(b) Find the fourth degree Taylor polynomial for $f(x) = \sec^2 x$ expanded around $x = \pi$.

(c) Find the fourth degree Taylor polynomial for $f(x) = \ln(\cos x)$ expanded around $x = \pi$.

8. (a) Simplify $(1 + i)^2/2$.

(b) Simplify $\sqrt{-2 + 6i}$. Express your answer in the form $a + bi$.

(c) If $z = 2e^{i\pi/2}$, what is the polar representation of $\bar{z}$?

(d) Find all solutions of the equation $y^3 = -1$.

9. Find two power series solutions for $y'' + xy' + y = 0$ and find the radius of convergence for each series.

10. Ol' Baldy appeared to lose 20 years after he bought his new toupee. Since Ol' Baldy received so many compliments on his youthful looks, he purchased a wool toupee to keep his head warm during the winter. Unfortunately for Ol'Baldy, a hungry moth found the toupee. The moth ate 50% of what remained of the hairpiece each day. Unfortunately for the moth, toupees lack nutrients and it died of malnutrition after ten full days. Originally, the wool toupee weighed 60 grams.

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10. (a) If the moth could eat forever at the same rate, write the total amount which could be eaten as a geometric series.

    (b) Subtract another geometric series to determine how much of the wig was eaten.

ANSWERS TO CHAPTER TEST

1. (a) False; let $a_i = 1$ and $b_i = -1$ for all i.

   (b) True

   (c) True

   (d) True

   (e) False; the signs do not alternate.

2. (a) $4x^3 - 9x^2 + 8x - 2$

   (b) $4(x - 1)^3 + 3(x - 1)^2 + 2(x - 1) + 1$

   (c) $4(x - 3/4)^3 + (37/8)(x - 3/4) + 59/8$

3. (a) $-2 < x < 0$

   (b) $-2 < x < 0$

   (c) $x < -2$ and $x > 0$

4. (a) Diverges; compare to $\Sigma_{n=1}^{\infty}(1/9n)$

   (b) Converges absolutely; $\Sigma_{n=1}^{\infty}(1/n^2)$ is a p-series

5. (a) $c_1 \sin 2x + c_2 \cos 2x + x \sin 2x/4 + e^x/5$

   (b) $c_1e^x + c_2(x - 1)e^x + c_3 = C_1e^x + c_2xe^x + c_3$

6. (a) No information

   (b) Converges absolutely

   (c) No information

   (d) Converges absolutely; note subscripts

   (e) Diverges

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7. (a) $(x - \pi) + (x - \pi)^3/3$

   (b) $1 + (x - \pi)^2 + 2(x - \pi)^4/3$

   (c) $(x - \pi)^2/2 + (x - \pi)^4/12$

8. (a) i

   (b) $a + bi$, where either $a = \sqrt{-1 + \sqrt{10}}$ and $b = \sqrt{1 + \sqrt{10}}$ or $a = -\sqrt{-1 + \sqrt{10}}$ and $b = -\sqrt{1 + \sqrt{10}}$

   (c) $2e^{-i\pi/2}$

   (d) $e^{-i\pi/3}$, $e^{i\pi/3}$, $e^{i\pi}$

9. $a_0\Sigma_{n=0}^{\infty}[(-1)^n x^{2n}/2^n n!]$ has radius of convergence = ∞ ;

   $a_1\Sigma_{n=0}^{\infty}[(-1)^{n+1}2^n(n!)x^{2n+1}/(2n + 1)!]$ has radius of convergence = ∞ .

10. (a) $\Sigma_{i=1}^{\infty} 60(1/2)^i$

    (b) $\Sigma_{i=1}^{\infty} 60(1/2)^i - \Sigma_{i=11}^{\infty} 60(1/2)^i = 60 - 15/256 = 15345/256$ grams .

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COMPREHENSIVE TEST FOR CHAPTERS 7-12 (Time limit: 3 hours)

1. True or false. If false, explain why.

   (a) The imaginary part of a complex number is a real number.

   (b) By l'Hôpital's rule, we have $\lim_{x\to 0}(\sin x/e^x) = \lim_{x\to 0}(\cos x/e^x) = 1$.

   (c) Every infinite series must either converge or diverge, but they can not do both.

   (d) An improper integral may have an upper sum and a lower sum, yet diverge.

   (e) One form of the formula for integration by parts is $\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$.

   (f) The Taylor series approximation of $e^{-(x^2)}$ for x near zero is $\Sigma_{i=0}^{\infty}(x^{-2i}/(2i)!)$.

   (g) Using an even number of subintervals, Simpson's method gives a better approximation than the trapezoidal method.

   (h) The center of mass of a region may lie on the region's perimeter, assuming uniform density.

   (i) If $f(t) > g(t)$ for all t , then the arc length of $f(t)$ is greater than that of $g(t)$ over the same interval.

   (j) The solution of $x'(t) = -3x(t)$ ia a decreasing function throughout its entire domain.

2. Multiple choice. (Choose the best answer.)

   (a) Which of the following is a solution of $y'' + 4y = 0$?

   (i) $\Sigma_{i=0}^{\infty}[(-1)^i x^{2i}/(2i)!]$

   (ii) $\Sigma_{i=0}^{\infty}[(-1)^i (2x)^{2i}/(2i)!]$

   (iii) $\Sigma_{i=0}^{\infty}[(-2)^i x^{2i}/(2i)!]$

   (iv) $\Sigma_{i=0}^{\infty}[(-1)^i 2x^{2i+1}/(2i+1)!]$

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2. (b) L'Hôpital's rule may be used for all of the following except:

(i) $\lim_{t\to 0^+} (\sin t)(\ln t)$

(ii) $\lim_{x\to 3} [2/(x^2 - 9) - \sqrt{3x^3}/(x^3 - 27)]$

(iii) $\lim_{n\to\infty} (1/n)^{3n+2}$

(iv) None of the above

(c) If one wishes to integrate $g(y) = (y^2 + 3y - 2)/y^2(y^2 + 3)^2(y - 1)$ by the method of partial fractions, the integrand should have the form:

(i) $(Ay + B)/y^2 + (Cy + D)/(y^2 + 3) + (Ey + F)/(y^2 + 3)^2 + G/(y - 1)$

(ii) $(Ay + B)/(y^2 + 3y - 2) + C/y^2 + (Dy + E)/(y^2 + 3)^2 + F/(y - 1)$

(iii) $A/y + B/y^2 + (Cy + D)/(y^2 + 3) + (Ey + F)/(y^2 + 3)^2 + G/(y - 1)$

(iv) None of the above

(d) The complex conjugate of $1 + i$ has polar representation

(i) $\sqrt{2}e^{-i\pi/4}$

(ii) $\sqrt{2}\exp(-\pi/4)$

(iii) $\sqrt{2}\, e^{\pi i/4}$

(iv) $\exp(-i\pi/4)$

(e) The mean value theorem states that for some point x_0 in (a,b) :

(i) $f(x_0) = [1/(b - a)]\int_a^b f'(x)dx$

(ii) $f(b - a) = [1/(b - a)]\int_a^b f(x_0)dx$

(iii) $f[(b + a)/2] = \int_a^b f(x_0)dx$

(iv) $f(x_0) = \int_a^b f(x)dx/(b - a)$

3. Short answers

(a) For Taylor's theorem, what is the remainder in the derivative form?

(b) State the ε-δ definition of the limit.

(c) Define $\sinh x$ in terms of exponentials.

(d) Express the arc length of $f(x) = x^n$ on $[0,1]$ as an integral.

(e) State the alternating series test.

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4. Numerical calculations

(a) Estimate $y(1)$ if $(y')^2 - xy = 0$ and $y(0) = 2$. Use a four-step Euler method.

(b) Use Simpson's method with $n = 4$ to estimate $\int_0^4 \exp(-x^2)dx$.

(c) State the formula used in Newton's method and use it to solve $x^3 + x^2 = -7$.

5. Fill in the blanks.

(a) According to the ratio test, a series $\Sigma_{n=0}^{\infty} a_n$ converges if _____.

(b) If $(-b \pm \sqrt{b^2 - 4ac})/2a$ equals r_1 and r_2, two distinct real roots, then the solution of $ay'' + by' + cy = 0$ is _____.

(c) The radius of convergence of $\Sigma_{i=0}^{\infty}(ix^i/3^i)$ is _____.

(d) The integration of $\cos^2 x$ is based upon the trigonometric identity $\cos^2 x =$ _____.

(e) The Maclaurin expansion of $1/(1 - x)$ is _____.

6. Integration problems. Evaluate the integrals.

(a) $\int(t^2 + 4t + 7)^{-1/2}dt$

(b) $\int(t^3 + 2t^2 + t)^{-1}dt$

(c) $\int(\Sigma_{n=2}^{\infty}(x^n/n^{8/3}))dx$

(d) $\int\cos^2 y \sin^3 y\, dy$

7. Do the following converge absolutely, converge conditionally, or diverge? Explain why.

(a) $\Sigma_{n=4}^{\infty}[n^2(n - 1)!/(2n + 1)!]$

(b) $4 - 1 + 1/4 - 1/16 + 1/64 - \ldots$

(c) $\int_{-2}^{10}(3e/x \ln|x|)dx$

(d) $\Sigma_{n=1}^{\infty}(3n/n^n)$

(e) $\Sigma_{n=10}^{\infty}[(4n + 5)(5n)/(3n + \sqrt{n})(n^2/5 - 1)](-1)^n$

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8. (a) What is the Taylor series expansion of e^{2x} around $x = 2$?

   (b) Use your answer in (a) to estimate e^1 . Use a fourth-order approximation.

   (c) How accurate is your answer in (b)?

9. Miscellaneous calculations.

   (a) Compute $\lim_{x\to 0}[(e^{1/x} - e^{-1/x})/x^2]$.

   (b) Find the general solution of $y'' - y' - 2y = x^2 + e$

   (c) Compute $\lim_{x\to 3+}[(x^2 - 9)/|3 - x|]$.

10. A former mathematician has given up his career to open a bakery. Cake flower decorations have green leaves which can be described in polar coordinates by $r = \cos 3\theta$. He uses red frosting to outline the leaves.

    (a) Write a formula to describe how much red frosting he uses to outline the leaves on each flower.

    (b) How much green frosting does he use for each flower's leaves?

ANSWERS TO COMPREHENSIVE TEST

1. (a) True

   (b) False; $(\sin x/e^x)$ has the form $0/1$, so l'Hôpital's rule can't be used.

   (c) True

   (d) True; consider $\int_0^\infty \sin x\, dx$.

   (e) True

   (f) False; it should be $\Sigma_{n=0}^{\infty}[(-x^2)^n/n!] = \Sigma_{n=0}^{\infty}[(-1)^n x^{2n}/n!]$.

   (g) True

   (h) True; consider the graph of $r = |\cos\theta|$ in polar coordinates.

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1. (i) False; let $f(t) = 10$ and $g(t) = t$ on $[0,1]$.

   (j) True

2. (a) ii

   (b) iv

   (c) iii

   (d) i

   (e) iv

3. (a) $[f^{(n+1)}(c)](x - x_0)^{n+1}/(n + 1)!$

   (b) $\lim_{x \to x_0} f(x) = L$ if $|f(x) - L| < \varepsilon$ whenever $|x - x_0| < \delta$.

   (c) $\sinh x = (e^{x} - e^{-x})/2$

   (d) $\int_0^1 \sqrt{1 + n^2 x^{2n-2}}\, dx$

   (e) An alternating series converges if (i) the signs are alternating, (ii) the terms are decreasing, and (iii) the limit of the terms is zero.

4. (a) 2.78

   (b) 0.836

   (c) $x_{n+1} = x_n - f(x_n)/f'(x_n)$; -2.31

5. (a) $\lim_{i \to \infty} |a_i/a_{i-1}| < 1$

   (b) $c_1 \exp(r_1 x) + c_2 \exp(r_2 x)$

   (c) 3

   (d) $(1 + \cos 2x)/2$

   (e) $\Sigma_{i=0}^{\infty} x^i$

6. (a) $\ln|\sqrt{t^2 + 4t + 7} + t + 2| + C$

   (b) $\ln|t/(t + 1)| + (t + 1)^{-1} + C$

   (c) $\Sigma_{n=2}^{\infty}[x^{n+1}/(n + 1)n^{8/3}]$

   (d) $\cos^5 y/5 - \cos^3 y/3 + C$

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7. (a) Converges absolutely; use ratio test.

   (b) Converges absolutely; it's a geometric series.

   (c) Diverges; $\int(3e/x \ln|x|) = 3e[\ln(\ln|x|)]$, which diverges at $x = 0$.

   (d) Converges absolutely; use root test.

   (e) Converges conditionally; use alternating series test and comparison test.

8. (a) $\Sigma_{n=0}^{\infty}[2^n e^4 (x - 2)^n/n!]$

   (b) $e^1 \approx e^4 - 3e^4 + 9e^4/2 - 9e^4/2 + 27e^4/8 = 11e^4/8$

   (c) Error $< 81e^4/40$

9. (a) Does not exist

   (b) $c_1\exp(2x) + c_2\exp(-x) - x^2/2 + x/2 - e/2 - 3/4$

   (c) 6

10. (a) $6\int_0^{\pi/6}\sqrt{1 + 8 \sin^2 3\theta}\ d\theta$

    (b) $\pi/4$

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